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1952

*Differential equations which can be factorized by the
Schrödinger-Infeld method*

A. F. Stevenson and W. A. Bassali 385

Measure in semigroups

B. R. Gelbaum and G. K. Kalisch 396

A coefficient problem for functions regular in an annulus

M. S. Robertson 407

*Analytic functions with an irregular linearly
measurable set of singular points*

I. E. Glover 424

*A new representation and inversion theory for
the Laplace transformation*

P. G. Rooney 436

A note on divergent series

G. M. Petersen 445

Multiply subadditive functions

G. G. Lorentz 455

The supremum of a family of additive functions

Israel Halperin 463

Map-colour theorems

G. A. Dirac 480

On the subalgebras of finite division algebras

J. L. Zemmer, Jr. 491

On D. E. Littlewood's algebra of S-functions

D. G. Duncan 504

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ON THE POSSIBLE FORMS OF DIFFERENTIAL EQUATION WHICH CAN BE FACTORIZED BY THE SCHRÖDINGER-INFELD METHOD

A. F. STEVENSON AND W. A. BASSALI

1. Introduction. The factorization method, initiated by Schrödinger [4] and modified and developed by Infeld [2], Duff [1], and Infeld and Hull [3], furnishes an elegant method of solving eigenvalue problems associated with certain ordinary differential equations of the second order. Not only the eigenvalues and eigenfunctions can thus be obtained, but also certain matrix elements associated with the eigenfunctions. Even if the method cannot be applied directly to eigenvalue problems, the factorization of an equation may still be of interest, since recurrence formulae may thus be established, e.g. for Bessel functions [3]. The connection of the method with Truesdell's [5] method of the "F-equation" has been discussed by Duff [1].

It is therefore of interest to give explicit forms to those differential equations which can be factorized. In the Infeld form of the factorization procedure—which is the only one we shall consider—this is equivalent to finding the solution of a certain differential-difference equation.

2. The form of $k_m(x)$. Infeld's form of the factorization procedure is as follows. The differential equation is written in the form

$$(2.1) \quad y'' + [r(x, m) + \lambda]y = 0,$$

where m is a parameter which can vary continuously. The equation (2.1) can then (by definition) be factorized if and only if it can be written in the two equivalent forms

$$(2.2) \quad \left[\frac{d}{dx} - k(x, m) \right] \left[\frac{d}{dx} + k(x, m) \right] y = [L(m) - \lambda]y,$$

$$(2.3) \quad \left[\frac{d}{dx} + k(x, m+1) \right] \left[\frac{d}{dx} - k(x, m+1) \right] y = [L(m+1) - \lambda]y,$$

where $L(m)$ is some function of m . The necessary and sufficient condition that (2.2) and (2.3) shall give the same differential equation (2.1) is easily found to be

$$(2.4) \quad k_{m+1}^2 - k_m^2 + k'_{m+1} + k'_m = L_m - L_{m+1},$$

where we write for brevity $k_m = k(x, m)$, $L_m = L(m)$, and the dashes stand for differentiation with respect to x . The function k_m must therefore satisfy the differential-difference equation

$$(2.5) \quad \frac{d}{dx} (k_{m+1}^2 - k_m^2 + k'_{m+1} + k'_m) = 0.$$

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Having found k_m from (2.5), L_m is given (to within an additive constant) from (2.4), and the function $r(x, m)$ in (2.1) is then given by

$$(2.6) \quad r(x, m) = k'_m - k_m^2 - L_m.$$

Thus all differential equations which can be factorized can be found if we can find all solutions of (2.5). We are only interested in solutions¹ which are continuous functions of m and x .

Particular solutions of (2.5) have been found by Infeld and Hull [3] and Duff [1] under the assumption that k_m is the sum of a finite number of positive and negative powers of m whose coefficients are functions of x . We shall solve (2.5) under a much more general assumption, namely that the differential equation (2.1) can be transformed into one with rational coefficients. To be more precise, our assumption is the following: *It is possible to find a transformation to new variables z, t defined by*

$$(2.7) \quad \frac{dt}{dx} = f(t), \quad y = g(t)z,$$

where $f(t)$ is independent of m , such that the equation (2.1), when multiplied by an appropriate function of t , becomes, for all λ and m , an equation whose coefficients are rational functions of t .

While we do not thus find the most general solution of (2.5) (which seems to be quite hard to find), our solution is nevertheless one of considerable generality, since all equations of interest in applications up to the present appear to be transformable in the above way, though the assumption that $f(t)$ is independent of m imposes a certain additional restriction.

Substituting (2.7) in (2.2), which by supposition is equivalent to (2.1), we find that (2.1) becomes

$$f^2 g \ddot{z} + (2f^2 \dot{g} + f \ddot{f} g) \dot{z} + [f^2 \ddot{g} + f \dot{f} \dot{g} + g(f \dot{k}_m - k_m^2 - L_m + \lambda)] z = 0,$$

where the dots denote differentiation with respect to t . According to our assumption, the ratios of the coefficients of \ddot{z} , \dot{z} , z , λz are rational functions of t . Using the fact that the sum, product, etc. of two rational functions is also a rational function, and that the derivative of a rational function is also rational, it is then easily shown that the functions f^2 , $f \dot{k}_m - k_m^2$ are rational functions of t . Similarly, using (2.3), we find that $f \dot{k}_{m+1} + k_{m+1}^2$, and hence also $f \dot{k}_m + k_m^2$, is a rational function. Hence f^2 , $f \dot{k}_m$, k_m^2 are rational functions, from which it follows that k_m/f is also rational. Thus we have

$$(2.8) \quad f(t) = [R(t)]^{\frac{1}{2}}, \quad k_m(t) = R_m(t)[R(t)]^{\frac{1}{2}},$$

where $R(t)$, $R_m(t)$ are rational functions of t , of which $R(t)$ is independent of m .

¹The whole factorization procedure becomes quite trivial if solutions which are discontinuous in m are allowed. We may, for instance, define $k(x, m)$ arbitrarily for $0 < m < 1$. By successive use of (2.5) with different values of m , we may then determine a function $k(x, m)$ which formally satisfies (2.5), but which is, in general, discontinuous for $m = 0, \pm 1, \pm 2, \dots$. Such solutions are, of course, of no use in eigenvalue problems.

Using (2.8) and the first of (2.7), (2.5) becomes

$$(2.9) \quad \frac{d}{dt} \left[R_{m+1}^2 - R_m^2 + \dot{R}_{m+1} + \dot{R}_m + \frac{\dot{R}}{2R} (R_{m+1} + R_m) \right] = 0.$$

We now consider the solution of (2.9) in the neighbourhood of the singularities of $R_m(t)$, which can only be poles. Let us first see if these poles can depend on m . The poles of $R_m(t)$ are given by the roots of a polynomial equation in t , say

$$(2.9a) \quad t^N + a_1(m)t^{N-1} + \dots + a_N(m) = 0.$$

Since $R_m(t)$ is a continuous function of m , the degree N of (2.9a) is independent of m . Let the distinct roots of (2.9a) be $t_1(m), \dots, t_M(m)$, where $M \leq N$. We may assume that these roots remain distinct from each other as m varies, except possibly for special values of m .

Now by considering the solution of (2.9) in the neighbourhood of a pole of $R_m(t)$, it is easily seen that at least one of $R_{m+1}(t)$ or $R_{m-1}(t)$ must have a pole of the same order at the same point. Hence each of the functions $t_i(m)$ must be equal either to one of the $t_i(m+1)$ or to one of the $t_i(m-1)$. Suppose that M' ($0 < M' < M$) of the $t_i(m)$ are not equal to any of the $t_i(m+1)$, but that each of the remaining $M - M'$ of the $t_i(m)$ are equal to one of the $t_i(m+1)$. Then there are M' of the $t_i(m+1)$ which are not equal to any of the $t_i(m)$; each of them must therefore be equal to one of the $t_i(m+2)$. Also, since $M - M'$ of the $t_i(m)$ are equal to one of the $t_i(m+1)$, each of the remaining $M - M'$ of the $t_i(m+1)$ must be equal to one of the $t_i(m+2)$. Hence every $t_i(m+1)$ ($i = 1, \dots, M$) is equal to one of the $t_i(m+2)$, and M' must be zero.

Thus when m is changed into $m+1$, the roots of (2.9a) can only undergo a permutation among themselves, multiplicities being preserved. Hence any symmetric function of the roots of (2.9a) is unaltered when m is changed into $m+1$. The same is therefore true of the coefficients $a_1(m), \dots, a_N(m)$ occurring in (2.9a). Thus these coefficients, and hence also the poles of $R_m(t)$, are periodic functions of m with period 1.

In what follows we shall, for simplicity, assume that m is an integer, as is indeed assumed by the authors previously mentioned. There is no loss of generality in this, since only values of m which differ by an integer occur in (2.5), and since if $k(x, m)$ is any solution of (2.5) then $k(x, m+c)$ is also a solution, where c is an arbitrary constant (similar remarks apply to (2.9) and its solutions $R(x, m)$). Our final results will hold for arbitrary m , provided arbitrary constants are replaced by arbitrary continuous functions of m of period 1 and the function $(-1)^m$ which occurs in some of the solutions is replaced by $e^{im\pi}$. Our solutions are then continuous functions of m , as required. We may now, according to what has been shown above, treat the poles of $R_m(t)$ as being independent of m .

Suppose, then, that in the neighbourhood of any pole of $R_m(t)$ in the finite part of the t -plane, say $t = t_1$, we have the expansions

$$(2.10) \quad R_m(t) = \sum_{s=0}^{\infty} a_s(m)(t-t_1)^{-s+1}, \quad \frac{\dot{R}}{2R} = \sum_{s=0}^{\infty} b_s(t-t_1)^{-1+s},$$

where $p \geq 1$, $a_0(m) \neq 0$ (b_0 may be zero). Let us, as usual, denote by Δ the difference operator ($\Delta u_m = u_{m+1} - u_m$), and let us put $\Delta' = \Delta + 2$. Substituting (2.10) in (2.9) and equating to zero the coefficients of the p lowest powers of $(t-t_1)$, we obtain

$$(2.11) \quad \begin{aligned} \Delta \left(\sum_{s=0}^p a_s a_{s-1} \right) &= 0, & s &= 0, 1, \dots, p-2, \\ \Delta \left(\sum_{s=0}^{p-1} a_s a_{p-1-s} \right) + (b_0 - p) \Delta' a_0 &= 0. \end{aligned}$$

If $p = 1$, only the second of (2.11) applies. Since $\Delta(a_0^2) = \Delta a_0 \cdot \Delta' a_0$, we see that these equations, solved in succession for a_0, a_1, \dots, a_{p-1} , give: either

$$a_s(m) = c_s (-1)^m, \quad s = 0, 1, \dots, p-1,$$

or

$$\begin{cases} a_s(m) = c_s, & s = 0, 1, \dots, p-2, \\ a_{p-1}(m) = c m + c', \end{cases}$$

where c_s, c, c' are arbitrary constants independent of m . Thus the coefficients of the negative powers of $(t-t_1)$ in the expansion of $R_m(t)$ in the neighbourhood of the pole $t = t_1$ are either linear functions of m or proportional to $(-1)^m$. If $t = \infty$ is a pole of $R_m(t)$, a similar investigation shows that the coefficients of the positive powers of t in the expansion of $R_m(t)$ in the neighbourhood of $t = \infty$ are also either linear functions of m or proportional to $(-1)^m$. We see, then, that the function $\Delta^2 \Delta' R_m(t)$ is an analytic function everywhere in the t -plane, which must, therefore, be a constant. Hence

$$(2.12) \quad \Delta^2 \Delta' R_m(t) = \lambda_m.$$

Regarding (2.12) as a difference equation for $R_m(t)$, the general solution can be written

$$R_m(t) = A_1(t) + m B_1(t) + (-1)^m C_1(t) + \mu_m,$$

where μ_m is a particular integral of (2.12) which is a function of m only. From (2.8), and returning to the variable x , we see therefore that $k_m(x)$ must be of the form

$$(2.13) \quad k_m(x) = A(x) + m B(x) + (-1)^m C(x) + \mu_m D(x).$$

3. The various types. It remains to find the possible forms of the functions $A(x), \dots, D(x)$ in (2.13). If (2.13) be substituted in (2.5), we see that, regarded as an equation in m , it is a linear relation between the following 11 functions of m :

$$(3.1) \quad \begin{aligned} &\mu_{m+1}^2 - \mu_m^2, \quad m \mu_{m+1}, \quad (-1)^m \mu_{m+1}, \quad \mu_{m+1}, \\ &m \mu_m, \quad (-1)^m \mu_m, \quad \mu_m, \quad m, \quad m(-1)^m, \quad (-1)^m, \quad 1. \end{aligned}$$

If these functions are all linearly independent (for integral values of m), then the coefficient of each of them can be equated to zero in (2.5). This gives $D = \text{constant}$, and μ_m is arbitrary. In the contrary case, there must be one or more linear relations between the functions (3.1), and hence μ_m is not arbitrary.

The first function, $\mu_{m+1}^2 - \mu_m^2$, in (3.1) may be linearly independent of the others. In this case, (2.5) gives again $D = \text{constant}$. In the contrary case, $\mu_{m+1}^2 - \mu_m^2$ is expressible as a linear function of the other 10 functions in (3.1), and (2.5) then becomes a linear relation between these ten functions (whose coefficients are functions of x). These ten functions cannot be linearly independent, since otherwise μ_m would be arbitrary. There must therefore exist at least one linear relation between them. If this relation involves μ_m but not μ_{m+1} (or vice versa), it gives

$$(3.2) \quad \mu_m = \frac{a + m b + (-1)^m c + m(-1)^m d}{a' + m b' + (-1)^m c'},$$

where a, \dots, c' are constants independent of m . If, on the other hand, the relation involves both μ_m and μ_{m+1} , it can be written in the form

$$(3.3) \quad \mu_{m+1} = \frac{a_1 + m b_1 + (-1)^m c_1}{a_3 + m b_3 + (-1)^m c_3} \mu_m + \frac{a_2 + m b_2 + (-1)^m c_2 + m(-1)^m d_2}{a_3 + m b_3 + (-1)^m c_3}.$$

Substituting for μ_{m+1} from (3.3) in (2.5), after having eliminated $\mu_{m+1}^2 - \mu_m^2$ in the manner indicated, it becomes a linear relation between the functions

$$m^2 \mu_m, \quad m^2 (-1)^m \mu_m, \quad m \mu_m, \quad (-1)^m \mu_m, \quad \mu_m, \\ m^2, \quad m^2 (-1)^m, \quad m (-1)^m, \quad (-1)^m, \quad m, \quad 1.$$

These functions again cannot be linearly independent. Hence we must have

$$(3.4) \quad \mu_m = \frac{c_1 m^2 + c_2 m^2 (-1)^m + c_3 m + c_4 m (-1)^m + c_5 (-1)^m + c_6}{c'_1 m^2 + c'_2 m^2 (-1)^m + c'_3 m + c'_4 (-1)^m + c'_5}.$$

Since (3.2) is included in (3.4), we see that (3.4) is the most general form possible for μ_m (unless μ_m is arbitrary).

We now write $(m-1)$ in place of m in (2.5) and subtract this from the original (2.5). Writing first $2m$, and then $2m-1$, in place of m in the resulting equation, we obtain the two equations

$$(3.5) \quad \frac{d}{dx} \left[(\xi_m^2 - \xi_{m-1}^2) D^2 + (f_m \xi_m - f_{m-1} \xi_{m-1}) + (f_m^2 - f_{m+1}^2) \right. \\ \left. + (\xi_m + 2\eta_m + \xi_{m-1}) D' \right] = 0,$$

$$(3.6) \quad \frac{d}{dx} \left[(\eta_m^2 - \eta_{m-1}^2) D^2 + (g_m \eta_m - g_{m-1} \eta_{m-1}) + (g_m^2 - g_{m-1}^2) \right. \\ \left. + (\eta_m + 2\xi_m + \eta_{m-1}) D' \right] = 0,$$

where

$$(3.7) \quad \xi_m = \mu_{2m+1} = \frac{am^2 + bm + c}{am^2 + \beta m + \gamma}, \quad \eta_m = \mu_{2m} = \frac{a'm^2 + b'm + c'}{a'm^2 + \beta'm + \gamma'},$$

$$f_m = (A + B - C) + 2mB, \quad g_m = (A + C) + 2mB.$$

The constants a, \dots, γ' in (3.7) are related to the constants c_1, \dots, c'_s occurring in (3.4). From (3.7) we note that

$$(3.8) \quad \mu_m = \frac{1}{2} [\xi_{\frac{1}{2}(m-1)} + \eta_{\frac{1}{2}m}] - \frac{1}{2} (-1)^m [\xi_{\frac{1}{2}(m-1)} - \eta_{\frac{1}{2}m}].$$

We suppose that ξ_m, η_m in (3.7) are expressed in their lowest terms.

If $a = \beta = a' = \beta' = 0$, then ξ_m, η_m are polynomials in m of degree two (or less). Leaving this case aside for the moment, at least one of ξ_m, η_m , say ξ_m , must have a pole of at least the first order at some point in the complex m -plane, say $m = m_0$. Then $\xi_m^2 - \xi_{m-1}^2$ has at least poles of the second order at $m = m_0$ and $m = m_0 + 1$. On the other hand, η_m can have at most a pole of the second order at one of these points. It then follows that $\xi_m^2 - \xi_{m-1}^2$ is linearly independent of all the other functions of m occurring in (3.5).² This equation therefore requires that D is constant. Similarly, if η_m has a pole, (3.6) shows that D is constant.

There remains to be considered the case where ξ_m, η_m are at most quadratics in m , say

$$(3.9) \quad \xi_m = am^2 + bm + c, \quad \eta_m = a'm^2 + b'm + c'.$$

Equating to zero the coefficients of m^3 in (3.6), (3.5), we then find that either D is constant or $a = a' = 0$. In the latter case we have, from (3.9) and (3.8),

$$\mu_m = am + \beta + (-1)^m(\gamma m + \delta),$$

where α, \dots, δ are constants. From (2.13), $k_m(x)$ can then be written

$$(3.10) \quad k_m(x) = A(x) + mB(x) + (-1)^m C(x) + m(-1)^m D(x).$$

Apart from this case, which we leave aside for the moment, we have shown that we can take D to be constant without loss of generality, so that from (2.13) we can always write

$$(3.11) \quad k_m(x) = A(x) + mB(x) + (-1)^m C(x) + \mu_m.$$

It is now convenient to write (2.5) in the equivalent form

$$(3.12) \quad \frac{d}{dx} \left[\sum_{m=m_0}^m (k'_{m+1} + k'_m) + (k_{m+1}^2 - k_m^2) \right] = 0$$

obtained by summing (2.5) with respect to m from m_0 to m , where m_0 is any particular value of m .

²The relation (3.5) need only, according to our assumptions, be satisfied for integral values of m , but since, when cleared of fractions, it becomes a polynomial relation in m , it must actually be satisfied for all values of m . The same holds for (3.6).

Substituting (3.11) in (3.12) we get

$$(3.13) \quad 2\mu_{m+1}P' + Q' = 0,$$

where

$$P = (A + B) + mB - (-1)^m C,$$

$$Q = m^2 F + 2m(A' + AB + F) - 2m(-1)^m BC - 2(-1)^m C(A + B) - f(m_0, x),$$

$$F = B^2 + B',$$

and $f(m_0, x)$ is a function which need not be given explicitly. Differentiating (3.13), and eliminating μ_{m+1} between (3.13) and the equation thus obtained we have

$$(3.14) \quad P'Q'' - Q'P'' = 0.$$

In (3.14) only known functions of m occur, so that we can equate coefficients of independent functions of m separately to zero to get a number of differential equations for the functions A, B, C . We thus find

$$F'B'' - F''B' = 0,$$

$$(3.15) \quad F'A'' - F''A' + (A' + AB)'B'' - (A' + AB)''B' = 0,$$

$$F'C'' - F''C' + 2(BC)'B'' - 2(BC)''B' = 0,$$

obtained by equating to zero the respective coefficients of $m^3, m^2, m^1(-1)^m$ in (3.14). These determine B, A, C in succession.

Having determined the possible forms for A, B, C , the requirement that (2.4) must be satisfied identically in x then determines μ_m and L_m , and also restricts the arbitrary constants occurring in these forms.

First case. If B is not a constant, the solutions of (3.15) are:

$$B = a - b \tan(bx + c),$$

$$A = b_1 \tan(bx + c) + c_1 e^{ax} \sec(bx + c),$$

$$C = b_1 \cot(bx + c);$$

$$\text{or} \quad B = a + \frac{1}{x+b}, \quad A = \frac{b_1}{x+b} + \frac{c_1 e^{ax}}{x+b}, \quad C = b_2 x;$$

$$\text{or} \quad B = \frac{1}{x+b}, \quad A = \frac{b_1}{x+b} + c_1 x, \quad C = b_2 x.$$

Substitution in (2.4) now gives the following solutions, in which we give also the function $r(x, m)$ as given by (2.4):

$$\begin{aligned}
\text{I} \quad & \begin{cases} k(x, m) = -(m+a)b \tan(bx+c) - d/(m+a), \\ L(m) = (m+a)^2 b^2 - d^2/(m+a)^2, \\ r(x, m) = -(m+a)(m+a+1)b^2 \sec^2(bx+c) - 2bd \tan(bx+c); \end{cases} \\
\text{II} \quad & \begin{cases} k(x, m) = -(m+a)b \tan(bx+c) + d \sec(bx+c), \quad L(m) = (m+a)^2 b^2, \\ r(x, m) = -[(m+a)(m+a+1)b^2 + d^2] \sec^2(bx+c) \\ \quad + bd[2(m+a)+1] \sec(bx+c) \tan(bx+c); \end{cases} \\
\text{III} \quad & \begin{cases} k(x, m) = -(m+a)b \tan(bx+c) + (-1)^m d \cot(bx+c), \\ L(m) = (m+a)^2 b^2 + 2(-1)^m bd(m+a), \\ r(x, m) = -(m+a)(m+a+1)b^2 \sec^2(bx+c) \\ \quad - d[d + b(-1)^m] \operatorname{cosec}^2(bx+c) + d^2; \end{cases} \\
\text{IV} \quad & \begin{cases} k(x, m) = (m+a)/(x+b) - c/(m+a), \quad L(m) = -c^2/(m+a)^2, \\ r(x, m) = -(m+a)(m+a+1)/(x+b)^2 + 2c/(x+b); \end{cases} \\
\text{V} \quad & \begin{cases} k(x, m) = (m+a)/(x+b) - c(x+b), \quad L(m) = 4cm, \\ r(x, m) = -(m+a)(m+a+1)/(x+b)^2 - c^2(x+b)^2 \\ \quad - c[2(m-a)+1]; \end{cases} \\
\text{VI} \quad & \begin{cases} k(x, m) = (m+a)/(x+b) - (-1)^m c(x+b), \\ L(m) = 2(-1)^m c(m+a), \\ r(x, m) = -(m+a)(m+a+1)/(x+b)^2 - c^2(x+b)^2 - (-1)^m c. \end{cases}
\end{aligned}$$

Second case. If B is constant, which can without loss of generality be taken equal to zero, since we may absorb a constant in μ_m in (3.11), the equations (3.15) are then satisfied identically, but we can use the equations

$$(3.16) \quad A'''A' - A''^2 = 0, \quad A'''C' - A''C'' = 0$$

obtained by equating the coefficients of m , $m(-1)^m$ in (3.14) to zero. Solving (3.16) for A and C we get: either

$$\begin{aligned}
\text{or} \quad & \text{(i) } A = a + b e^{cx}, \quad C = a' + b' e^{cx}, \\
& \text{(ii) } A = a + bx, \quad C \text{ arbitrary.}
\end{aligned}$$

Substitution in (2.4) now gives the additional solutions:

$$\begin{aligned}
\text{VII} \quad & \begin{cases} k(x, m) = a + bm + ce^{-bx}, \\ r(x, m) = -c(2mb + 2a + b)e^{-bx} + ce^{-2bx}; \end{cases} \quad L(m) = -(a + bm)^2, \\
\text{VIII} \quad & \begin{cases} k(x, m) = (a + bx) + (-1)^m c/(a + bx), \quad L(m) = -2bm - 2c(-1)^m, \\ r(x, m) = -(a + bx)^2 - c[c + b(-1)^m]/(a + bx)^2 + (2m + 1)b; \end{cases} \\
\text{IX} \quad & \begin{cases} k(x, m) = f(m), \text{ where } f \text{ is an arbitrary function of } m, \quad L(m) = \{f(m)\}^2, \\ r(x, m) = 0; \end{cases} \\
\text{X} \quad & \begin{cases} k(x, m) = (-1)^m C(x), \text{ where } C \text{ is an arbitrary function of } x, \quad L(m) = 0, \\ r(x, m) = (-1)^m C'(x) - [C(x)]^2. \end{cases}
\end{aligned}$$

We now return to the case where $k_m(x)$ is given by (3.10). Substituting in (2.5), and equating to zero the coefficients of the various independent functions of m , we have

$$(3.17) \quad BD = c_1, \quad BC + AD = c_2, \quad B^2 + B' + D^2 = c_3, \quad 4AC + D' = c_4, \\ A' + AB + CD = c_5,$$

where c_1, \dots, c_5 are arbitrary constants. If $B = 0$ or $c_1 = 0$ we see that $D = \text{constant}$. If $B \neq 0$, $c_1 \neq 0$, it is possible to eliminate D, C, A', D' between the above equations, and obtain two equations between A and B which must be compatible. These equations are

$$4A^2 - 4\frac{c_2}{c_1}BA + \left(\frac{c_4}{c_1} - 1\right)B^2 - \frac{c_1^2}{B^2} + c_3 = 0, \\ 4\left[B - \frac{c_1^2}{B^2}\right]A^2 + 2\left(\frac{c_2 c_3}{c_1} - 2c_5 + \frac{2c_1 c_2}{B^2} - \frac{2c_2 B^2}{c_1}\right)A + 2\left(\frac{c_2 c_3}{c_1} B - \frac{c_2^2}{B}\right) \\ + \left(B^2 + \frac{c_1^2}{B^2} - c_3\right)\left[\left(\frac{c_4}{c_1} - 1\right)B + \frac{c_1^2}{B^2}\right] = 0.$$

They are compatible if

$$c_4 = c_2^2/c_1, \quad c_5 = c_2 c_3/2c_1,$$

where c_1, c_2, c_3 may be arbitrary.

The general solution of (3.17) can then be written

$$(3.18) \quad \begin{cases} \int \frac{B^2}{B^4 - c_3 B^2 + c_1^2} dB = -x + \text{constant}, \\ 4A^2 - 4\frac{c_2}{c_1}BA + \left(\frac{c_2^2}{c_1^2} - 1\right)B^2 - \frac{c_1^2}{B^2} + c_3 = 0, \\ D = c_1/B, \quad C = (c_2/B) - (c_1 A/B^2). \end{cases}$$

Thus we have the additional solution of (2.4):

$$\text{XI} \begin{cases} k_m(x) = A(x) + m B(x) + (-1)^m C(x) + m(-1)^m D(x), \\ L(m) = -[c_1 m^2 + (c_2 c_3/c_1)m + 2(-1)^m\{c_1 m^2 + c_2 m + c_2^2/4c_1\}], \\ r(x, m) = m(m+1)B' + (2m+1)A' + (-1)^m[(m+\frac{1}{2})D' + C'] \\ \quad \quad \quad - (A^2 + C^2), \end{cases}$$

where A, B, C, D are given by (3.18).

The above types I–XI exhaust all the possible solutions under our assumption. They include as special cases all those which have been given previously [1, 3]. Of these eleven types IX, X are trivial, though X is perhaps of some interest as it shows that any equation of the form

$$(3.19) \quad y'' + [r(x) + \lambda]y = 0,$$

can be formally factorized by the Infeld procedure, regarding it³ as a particular

³Such a device is termed "artificial factorization" by Infeld and Hull [3].

case of the more general form (2.1) for a particular value of m . For a function $C(x)$ can always be found such that

$$(3.20) \quad C'(x) - [C(x)]^2 = r(x),$$

and then (3.19) coincides with X for any even m .

Of the remaining types, V, VI, and VII lead to differential equations which can be made to coincide for special values of m by proper choice of the arbitrary constants or by absorbing constants that occur in $r(x, m)$ in the parameter λ . They thus represent different ways of factorizing the same differential equation, namely one of the type

$$(3.21) \quad y'' + \left[a(x+b)^2 + \frac{c}{(x+b)^2} + \lambda \right] y = 0.$$

Similarly, types II and III are essentially the same, as can be seen by writing $\frac{1}{2}(bx+c)$ instead of $(bx+c)$ in III.

On the other hand XI is an essentially new type, though it is doubtful if it can be applied to eigenvalue problems. The integral occurring in (3.18) can easily be evaluated, but it is not possible in general to express B as an explicit function of x in terms of standard functions.

It remains to be seen whether all these types can be transformed to differential equations with rational coefficients in accordance with our original assumption. This is not necessarily the case, since we have only deduced *necessary* conditions that the differential equation (2.1) shall be thus transformable. It is easily seen that the following substitutions suffice:

$$\text{For I and III:} \quad \tan(bx+c) = t, \quad dt/dx = b(1+t^2).$$

$$\text{For II:} \quad \sin(bx+c) = t, \quad dt/dx = b(1-t^2)^{\frac{1}{2}}.$$

$$\text{For VII:} \quad e^{-bx} = t, \quad dt/dx = -bt.$$

IV, V, VI, VIII, IX are already in the required form. It is evident that X cannot in general be so transformed, nor does it seem possible in general to transform XI. In the particular case in which A , as determined by the quadratic equation in (3.18), is a rational function of B (which occurs if $c_3 = \pm 2c_1$), it is possible to transform XI to a differential equation with rational coefficients by means of the substitution

$$B(x) = t, \quad \frac{dt}{dx} = c_3 - t^2 - \frac{c_1^2}{t^3}.$$

It will be seen that, in all the above cases, it is not necessary to change the dependent variable, as envisaged by (2.7), in order to effect the required transformation.

Apart from X and XI, all the differential equations, when reduced to forms with rational coefficients, are either of hypergeometric or confluent hypergeometric type.

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REFERENCES

1. G. F. D. Duff, *Factorization ladders and eigenfunctions*, Can. J. Math., vol. 1 (1949), 379-396.
2. L. Infeld, *On a new treatment of some eigenvalue problems*, Phys. Rev., vol. 59 (1941), 737-747.
3. L. Infeld and T. E. Hull, *The factorization method*, Rev. Mod. Phys., vol. 23 (1951), 21-68.
4. E. Schrödinger, Proc. Royal Irish Acad., vol. A46 (1940), 9.
5. C. A. Truesdell, *A unified theory of special functions* (Princeton, 1948).

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MEASURE IN SEMIGROUPS

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1. Introduction. The major portion of this paper is devoted to an investigation of the conditions which imply that a semigroup (no identity or commutativity assumed) with a bounded invariant measure is a group. We find in §3 that a weakened form of "shearing" is sufficient and a counter-example (§5) shows that "shearing" may not be dispensed with entirely. In §4 we discuss topological measures in locally compact semigroups and find that shearing may be dropped without affecting the results of the earlier sections (Theorem 2). The next two theorems show that under certain circumstances (shearing or commutativity) the topology of the semigroup (already known to be a group by virtue of earlier results) can be weakened so that the structure becomes a separated compact topological group. The last section treats the problem of extending an invariant measure on a commutative semigroup to an invariant measure on its quotient structure.

2. Measure-theoretic and topological preliminaries. We summarize in this section all definitions, concepts, and general conditions to which reference will be made in the remainder of the paper.

We shall be dealing with semigroups, denoted by S , in which there is a two-sided cancellation law. In general, commutativity and the existence of an identity will not be assumed, unless something to the contrary is stated. Without further comment we shall use the measure-theoretic notations and concepts of [2], such as ring, measure on a ring, outer measure, inner measure, completion of a measure, content, etc. We shall consider the preceding on both S and $S \times S$, and we shall distinguish between them by means of the subscripts 1 and 2 for S and $S \times S$ respectively.

A ring \mathfrak{R}_1 of subsets of S will be called *left-invariant* in case

$$(A) \quad x \in S, A \in \mathfrak{R}_1 \text{ imply } xA \in \mathfrak{R}_1.$$

Similarly, a measure m_1 on a left-invariant ring \mathfrak{R}_1 will be called *left-invariant* in case

$$(B) \quad x \in S, A \in \mathfrak{R}_1 \text{ imply } m_1(A) = m_1(xA).$$

We observe that the Conditions A and B for m_1 and \mathfrak{R}_1 imply the same for \bar{m}_1 , \mathfrak{E}_1 , and the corresponding entities of $S \times S$.

In $S \times S$ we shall encounter the following transformations:

$$\text{shearing, } \theta(x, y) = (x, xy);$$

$$\text{reflection, } \pi(x, y) = (y, x).$$

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We note that π is a measure preserving transformation for $m_1 \times m_1$ and such of its extensions as we shall have occasion to consider. Concerning θ we shall sometimes make the following assumption:

(C) $A \in \mathcal{R}_1, B \in \mathcal{R}_1$ imply $\theta(A \times B)$ is \bar{m}_2 -measurable, where \bar{m}_2 is the completion of $\bar{m}_1 \times \bar{m}_1$.

We shall be concerned principally with the case where

(D) $\sup (m_1(A): A \in \mathcal{R}_1) = 1$.

We note that this condition implies the same for \bar{m}_1, \bar{m}_2 , etc.

Part of our discussion will revolve around the following cases:

(E) S is locally compact and $T_1; xy$ is continuous on $S \times S$; $\mathcal{R}_1 = \mathcal{R}(\mathcal{C}_1)$, where \mathcal{C}_1 is the family of compact sets of S ; m_1 is regular on \mathcal{R}_1 .

We note that the assumption of regularity for m_1 is not restrictive. For, by [2], there exists a non-trivial regular Borel measure m'_1 associated with any non-trivial Borel measure m_1 such that $m'_1(C) > m_1(C)$ for all $C \in \mathcal{C}_1$.

(F) S is Abelian.

(G) $x \in S, A \subset S, xA \in \mathcal{R}_1$ imply $A \in \mathcal{R}_1$.

3. Semigroups with shearing. We shall be concerned with xS and $xS \times xS$ as well as with the original semigroup S and the rings and measures in the former, which are produced by translation, will be designated by $\mathcal{R}'_1, \dots, m'_1, \dots$, where \mathcal{R}'_1 , for instance, consists of all translates by x of the elements of \mathcal{R}_1 , and $m'_1(xA) = m_1(A)$, for A in \mathcal{R}_1 .

LEMMA 1. *Conditions (A B D) imply that $(x, x)E$ is \bar{m}'_2 -measurable if E is \bar{m}_2 -measurable, and that $\bar{m}'_2((x, x)E) = \bar{m}_2(E)$.*

Proof. The lemma is clearly true if E is the rectangular set $A \times B$, where $A \in \mathcal{R}_1, B \in \mathcal{R}_1$. If $\bar{m}_2(E) = 0$, then $\bar{m}'_2((x, x)E) = 0$. For, given an $\epsilon > 0$, one can find sequences of sets $A_n \in \mathcal{R}_1, B_n \in \mathcal{R}_1$ such that

$$\bigcup_{n=1}^{\infty} (A_n \times B_n) \supset E$$

and such that

$$\sum_{n=1}^{\infty} \bar{m}_2(A_n \times B_n) < \epsilon.$$

The assertion is now obvious. In the general case E may be expressed as a union of two sets M and N , where M is m_2 -measurable and $\bar{m}_2(N) = 0$. Clearly $(x, x)M$ is m'_2 -measurable, and the proof is complete.

THEOREM 1. *Conditions (A B C D) imply that S is a group.*

Proof. 1. We first show that S has an identity. Condition (D) implies that we may choose an $A \in \mathcal{R}_1$ such that $\bar{m}_2(A \times A) > 0.9$. Then

$$\bar{m}_2(\theta(A \times A)) = \int_A m_1(xA) dm_1(x) = (m_1(A))^2 > 0.9.$$

Obviously, $\bar{m}_2(\pi\theta(A \times A)) > 0.9$, whence, by (D), $\theta(A \times A) \cap \pi\theta(A \times A)$ is not empty. Hence there exist pairs (x, y) and (u, v) in $A \times A$ such that $(x, xy) = (uv, u)$, i.e., $x = uv$, $xy = u$, $xyv = uv = x$, i.e., yv is a right identity for x , and hence, by a standard algebraic result, a two-sided identity for S .

2. We show that $xS, \mathfrak{R}'_1, \dots, m'_1, \dots$, etc., constitute a system satisfying conditions (A B C D). The facts that xS is a semigroup, \mathfrak{R}'_1 is a ring with a measure m'_1 , as well as their fulfilling conditions (A B D) are easily verified. Condition (C) is verified as follows. Let $A \in \mathfrak{R}_1, B \in \mathfrak{R}_1$. Then

$$\theta(xA \times xB) = (x, x)\theta(A \times B),$$

which by Condition (C) (for S) and Lemma 1 is m'_2 -measurable. This shows that xS has an identity for all $x \in S$, and thus S is a group.

4. Locally compact semigroups. We shall show that if S is locally compact then, in a certain sense, condition (C) (shearing preserves measurability) can be dispensed with, while the conclusion of Theorem 1 remains undisturbed. Our proof will be based upon the fact that if m_1 is a Borel measure on S , then $m_1 \times m_1 = m_2$ can be extended to a Borel measure μ_2 on $S \times S$. This construction is carried out by means of a *partial content* which we shall now define. A partial content is a non-negative, finite-valued, finitely additive, and sub-additive monotone set function defined on a class of compact sets which has the following properties:

- (i) the union of any two elements of the class is in the class;
- (ii) if C is in the class, and if C is contained in the union of two open sets U and V , then there are two sets in the class, $D \subset U, E \subset V$, such that $C = D \cup E$. A partial content is very closely analogous to the content defined in [2, p. 231]. The development found in [2, §53] can be duplicated for a partial content, and one obtains a regular Borel measure μ induced by the given partial content. (Note that it is possible to follow the development of [2, §53] without restricting oneself to σ -bounded sets. Unless explicit indication to the contrary is made, we shall not restrict ourselves, in what follows, to the exclusive consideration of σ -bounded sets.)

LEMMA 2. *If m_1 is a Borel measure on a locally compact space X , then $m_1 \times m_1 = m_2$ can be extended to a Borel measure μ_2 on $X \times X$.*

Proof. We observe that the set function $m_2(C)$ on the class of m_2 -measurable compact sets of $X \times X$ is a partial content in the sense defined above, and the regular Borel measure μ_2 induced by it is easily seen to be an extension of m_2 on $X \times X$.

The measure μ_2 just defined is used in the proof of the following lemma which, as we shall show, can be used to avoid positing Condition (C) (shearing) in the presence of local compactness.

LEMMA 3. Conditions (A B D E) imply that $(\mu_2) * (\theta(S \times S)) = 1$.

Proof. Given $\epsilon > 0$, select a compact $C \subset S$ such that $m_1(C) > 1 - \epsilon$. Then $\theta(C \times C) = D$ is a compact subset of $S \times S$ (by E). Owing to the regularity of μ_2 and m_1 (hence of m_2 and \bar{m}_2), there exists a decreasing sequence of open sets U_n such that for all n ,

$$U_n \in \mathfrak{R}_1 \times \mathfrak{R}_1, U_n \supset D, m_2(U_n) \downarrow \mu_2(D).$$

Thus if

$$A = \bigcap_{n=1}^{\infty} U_n,$$

then $\bar{m}_2(A) = \mu_2(D)$. Since Fubini's theorem is applicable for the measure \bar{m}_2 ,

$$\bar{m}_2(A) = \int_S \bar{m}_1(A_x) d\bar{m}_1(x).$$

Clearly, $A_x \supset D_x$, whence, whenever A_x is \bar{m}_1 -measurable (which is true for almost every x), $\bar{m}_1(A_x) > m_1(D_x)$ (D_x is compact, hence m_1 -measurable for every x). But $D_x = xC$, for x in C , and is empty otherwise. Thus

$$\begin{aligned} \mu_2(D) = \bar{m}_2(A) &= \int_S \bar{m}_1(A_x) d\bar{m}_1(x) > \int_S m_1(D_x) d\bar{m}_1(x) = \int_C m_1(xC) d\bar{m}_1(x) \\ &= (m_1(C))^2 > (1 - \epsilon)^2. \end{aligned}$$

Since ϵ is arbitrary, the contention of the lemma follows.

THEOREM 2. Conditions (A B D E) imply that S is a group.

Proof. We show that S has an identity. Lemma 3 shows that $\theta(S \times S) \cap \pi\theta(S \times S)$ is not empty, and the existence of an identity then follows as in the proof of Theorem 1. The theorem will be proved if we show that xS has an identity for all $x \in S$. To this end, we select a compact set C for which $m_1(C) > 1 - \epsilon$. Then

$$\theta(xC \times xC) = (x, x)\theta(C \times xC)$$

is μ_2 -measurable. Reasoning similar to that found in the proof of Lemma 3 reveals that

$$\mu_2(\theta(xC \times xC)) \geq (1 - \epsilon)^2,$$

and we conclude that $(\mu_2) * (\theta(xS \times xS)) = 1$; hence $\theta(xS \times xS) \cap \pi\theta(xS \times xS)$ is not empty; xS has an identity, and the proof is complete.

If, in addition to the conditions of Theorem 2, S also satisfies Condition (F) (commutativity), we can strengthen our results. For these purposes we first prove the following purely measure-theoretic lemma.

LEMMA 4. Let X be a locally compact space, λ a content defined on the compact subsets of X , and μ the measure engendered by λ . Then every open subset of X is measurable.

Remarks. Note that we do not restrict ourselves to the σ -bounded sets. When only the σ -bounded sets are considered (as is the case in [2]), the conclusion of the lemma should be amended to read that every σ -bounded open subset of X is measurable. We note also that if a Borel measure is given, it is possible to find, via an appropriate content, an extension of the given measure, with respect to which every open set is measurable.

Proof. We first prove that if an open set has finite outer measure, then it is measurable. Thus let U be an open set such that $\mu^*(U)$ is finite. Let $C_n \subset U$ be compact sets such that

$$\lambda(C_n) \uparrow \lambda^*(U) = \mu^*(U).$$

Observe that $\lambda(C_n) \leq \mu(C_n) \leq \mu^*(U)$, whence $\mu(C_n) \uparrow \mu^*(U)$. Thus if

$$A = \bigcup_{n=1}^{\infty} C_n,$$

A is measurable and $\mu(A) = \mu^*(U)$. Applying the Carathéodory criterion to A , using the set U as the testing set, we find

$$\mu^*(U) \geq \mu^*(U \cap A) + \mu^*(U \cap A'),$$

where A' is the complement of A . Since $A \subset U$, and since A is measurable, the last inequality becomes $\mu^*(U) \geq \mu(A) + \mu^*(U \cap A')$. Thus, by the finiteness of $\mu^*(U) = \mu(A)$, it follows that $\mu^*(U \cap A') = 0$; hence $U \cap A'$ is measurable; and U , being the union of the measurable sets A and $U \cap A'$, is measurable.

Next, referring to [2, §53, Theorem D] (which is valid even when the σ -boundedness restrictions are removed), we must show that for an arbitrary open set V (which we may clearly assume to have finite outer measure)

$$(1) \quad \mu^*(V) = \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \cap U'),$$

where U is the open set whose measurability we seek to establish. Since $U \cap V$ is an open set of finite outer measure, it is measurable. Since V is measurable, $V \cap U'$, which is the relative complement of $U \cap V$ in V , is also measurable, whence (1) is established and the lemma follows.

THEOREM 3. *Conditions (A B D E F) imply that the topology of S may be weakened in such a way that S becomes a separated, compact topological group whose Haar measure coincides with the measure \bar{m}_1 .*

Proof. By Theorem 2 above and by [1, Theorem 9] there is a weakening of the topology of S which makes S a (separated) topological group S' . Local compactness of S' is a consequence of [1, Theorem 8] because, in the light of this result, S' is the continuous open image of the locally compact space $S \times S$.

Since S' is a continuous image of S , compact sets of S' are closed in S , and hence \bar{m}_1 -measurable by Lemma 4. This completes the proof of the theorem since a locally compact topological group with a bounded Haar measure is compact.

The above shows that the presence of commutativity (F) implies that the

topology of S may be weakened so that S becomes a separated, compact topological group. In what follows we shall show that we may replace the commutativity assumption by a shearing (C) assumption without sacrificing the conclusion.

LEMMA 5. *Let S be a group which satisfies (A B C E). Then the Weil topology for S is separated and is weaker than the given topology.*

Proof. (i) By virtue of the continuity of multiplication we see that if C is compact and if U is open, $U \supset C$, one can find a neighbourhood of the identity V for which $CV \subset U$: for each x in C , choose a neighbourhood of the identity W such that $xW^2 \subset U$. Then one can find a finite subset of the x 's and the associated W 's for which

$$C \subset \bigcup_{i=1}^n x_i W_i,$$

and if

$$V = \bigcap_{i=1}^n W_i,$$

then $CV \subset U$. Similarly, one can find a neighbourhood of the identity V' such that $V'C \subset U$.

(ii) Let $N(\epsilon; E) = \{x: \rho(xE, E) < \epsilon\}$ [2, p. 270]. Then, given two measurable sets A and B such that $A \subset B$, and given an $\epsilon > 2\bar{m}_1(B - A)$, there is a $\delta > 0$, such that $N(\delta; A) \subset N(\epsilon; B)$. From the identity

$$\begin{aligned} (A \triangle xA) \cup ((B - xB) - A) \cup ((xB - B) - xA) \\ = (B \triangle xB) \cup ((A - xA) \cap xB) \cup ((xA - A) \cap B), \end{aligned}$$

it follows that $\rho(B, xB) \leq \rho(A, xA) + 2\eta$, where $\eta > \bar{m}_1(B - A) > 0$. Thus if $\delta + 2\eta < \epsilon$, our conclusion follows. Thus if \mathfrak{F} is a family of sets (e.g., $\mathfrak{F} = \mathbb{C}_1$) such that for $\epsilon > 0$ and $A \in \mathfrak{R}_1$ there is an $F \in \mathfrak{F}$, $F \subset A$, such that $\bar{m}_1(A - F) < \epsilon$, the family $\{N(\epsilon; F)\}$ is a basis at the identity for the Weil topology.

(iii) By the use of the Fubini theorem, we see that the Weil topology may be constructed on the basis of our condition (C) which demands less than is demanded in [2, p. 257] ("shearing is a measurable transformation").

(iv) The Weil topology in our case is separated because the topology of S is separated and because we are dealing with a topological measure (Condition (E)).

(v) We now show that the Weil topology is weaker than the given one. Indeed, if $N(\epsilon; E)$ is given, there exists by virtue of (ii) a compact set $C \subset E$, and a positive δ such that $N(\delta; C) \subset N(\epsilon; E)$. As in (ii) we may choose δ and a positive η such that $\delta + 2\eta < \epsilon$. Choose an open set $U \supset C$ such that $\bar{m}_1(U - C) < \frac{1}{2}\delta$, and then by (i) find V such that $VC \subset U$. Then if $x \in V$,

$$\rho(C, xC) = \bar{m}_1(C - xC) + \bar{m}_1(xC - C) < \delta,$$

since $\bar{m}_1(U - xC) = \bar{m}_1(U - C)$ which is less than $\frac{1}{2}\delta$. Thus

$$V \subset N(\delta; C) \subset N(\epsilon; E).$$

THEOREM 4. *Conditions (A B C D E) imply that the topology of S may be weakened in such a way that S becomes a separated, compact topological group whose Haar measure coincides with the measure \bar{m}_1 .*

Proof. Theorem 1 or Theorem 2 implies that S is a group. Lemma 5 shows that the Weil topology for S is separated, and is weaker than the given topology. Owing to (D) (boundedness of the measure) we may employ the technique of [4, p. 38] to show that if S is a topological group with a bounded invariant topological measure, and if there exists a compact set of positive measure, then S is compact. Since S is of the character described in the preceding sentence, the theorem is proved.

Remarks. 1. It is clear from the results of Montgomery [3] that if a semigroup S satisfies the conditions of Theorem 2 as well as the second axiom of countability, then S is a compact topological group (it is regular, hence metrizable; the compactness follows from the boundedness of the Haar measure).

2. Completion regularity [2, p. 230] in $S \times S$ relative to m_2 , in addition to Conditions (A B D E), implies the conclusion of Theorem 4.

5. A counter-example. We now present a counter-example which shows that Condition (C) in the hypothesis of Theorem 1 cannot be completely eliminated. The semigroup S which we shall construct consists of the non-negative elements of the following ordered group G . Let A be a linearly ordered set which contains no countable co-final subset, e.g., the ordinals of the first and second classes. Let G be the group of all weak¹ mappings of A into the reals, where the mappings are linearly ordered as follows: let $x = x(a)$ and $y = y(a)$ be two distinct weak mappings, and let $a_0 = \sup\{a : x(a) \neq y(a)\}$ (which exists since the mappings are weak and A is linearly ordered). We say $x > y$ in case $x(a_0) > y(a_0)$. In this way, G becomes an ordered group when addition is defined vectorially. Let \mathcal{R}_1 be the ring generated by the bounded and unbounded, open, half-open, or closed intervals of S ; m_1 is defined to be zero for all bounded intervals and finite unions thereof and to be one for a finite union of intervals at least one of which is unbounded. One verifies easily that m_1 is a measure on the ring \mathcal{R}_1 , and that m_1 and \mathcal{R}_1 jointly satisfy (A B D).

6. Commutative semigroups. Thus far we have investigated semigroups with bounded invariant measures. We now turn to the consideration of commutative semigroups on which there are invariant, not necessarily bounded, measures. In the following we shall consider the Cartesian product $S \times S$, the equivalence relation R defined in $S \times S$ by: $(a, b)R(c, d)$ if and only if $ad = bc$, and the canonical mapping ϕ of $S \times S$ on the set of R -equivalence classes $Q(S) = S \times S/R = G$ (see [1]).

¹We say that a mapping $x(a)$ is weak in case $x(a) \neq 0$ for at most a finite number of a 's. The set of these mappings is sometimes called the weak direct product of the reals over the index set A .

THEOREM 5. *Conditions (A B F G) imply that there is in $G = Q(S)$ a translation invariant measure μ_1 . The measure induced by μ_1 on S considered as a subset of G coincides with the given measure.*

Proof. It is clear that the set $\phi(xS, x)$ is the same for all x and is a sub-semigroup of G isomorphic to S . Thus we shall always consider S as this sub-semigroup of G . Consider the family \mathfrak{A}_1 of subsets of G consisting of all sets of the form gE , where $g \in G$, and $E \in \mathfrak{R}_1$. We show that \mathfrak{A}_1 is a ring of sets in G (obviously invariant in G). Indeed let

$$g_i E_i \in \mathfrak{A}_1, \quad g_i = a_i b_i^{-1}, \quad a_i \in S, \quad b_i \in S \quad (i = 1, 2).$$

Then

$$g_1 E_1 \cup g_2 E_2 = (b_1 b_2)^{-1} (b_2 a_1 E_1 \cup b_1 a_2 E_2),$$

$$g_1 E_1 - g_2 E_2 = (b_1 b_2)^{-1} (b_2 a_1 E_1 - b_1 a_2 E_2),$$

whence \mathfrak{A}_1 is a ring. We now define an invariant measure μ_1 on \mathfrak{A}_1 : $\mu_1(gE) = m_1(E)$. This number is well defined, for if

$$g_1 E_1 = g_2 E_2, \quad g_1 = a_1 b_1^{-1}, \quad g_2 = a_2 b_2^{-1} \quad (a_i \text{ and } b_i \text{ in } S),$$

then

$$b_2 a_1 E_1 = b_1 a_2 E_2$$

whence

$$\mu_1(g_1 E_1) = m_1(E_1) = m_1(b_2 a_1 E_1) = m_1(b_1 a_2 E_2) = m_1(E_2) = \mu_1(g_2 E_2).$$

Now we show that μ_1 is an invariant measure on \mathfrak{A}_1 —and we need only verify the countable additivity of μ_1 . Let $g_i E_i$ be disjoint sets of \mathfrak{A}_1 whose union $g_0 E_0$ is also in \mathfrak{A}_1 , where $g_i = a_i b_i^{-1}$ (a_i and b_i in S). The sets $b_0 g_i E_i$ are disjoint sets whose union $a_0 E_0$ is in S ; hence each $b_0 g_i E_i$ is a subset of S which is also in \mathfrak{R}_1 : for, $b_0 g_i E_i = b_0 a_i E_i$ is in \mathfrak{R}_1 , and hence, by Condition (G), $b_0 g_i E_i$ is in \mathfrak{R}_1 too. Thus,

$$m_1(a_0 E_0) = \sum_{i=1}^{\infty} m_1(b_0 g_i E_i) = \sum_{i=1}^{\infty} m_1(E_i) = m_1(E_0),$$

whence $\mu_1(g_0 E_0) = \sum \mu_1(g_i E_i)$.

Our next objective is to show that the boundedness of the measure (Condition (D)) allows us to dispense with Condition (G) in the preceding theorem. To this end we prove a preliminary lemma.

LEMMA 6. *Conditions (A B D F) imply the truth of the following statements:*

(a) Let $\mathfrak{R}'_1 = \mathfrak{R}_1 \cup \{S - A : A \in \mathfrak{R}_1\}$; $m'_1(S - A) = 1 - m_1(A)$, $m'_1(A) = m_1(A)$, $A \in \mathfrak{R}_1$. \mathfrak{R}'_1 is a ring of sets; m'_1 a measure on \mathfrak{R}'_1 , and m'_1 and \mathfrak{R}'_1 satisfy Conditions (A) and (B).

(b) Let $\mathfrak{R}''_1 = \{E : E \subset S; \text{ for some } x \in S, xE \in \mathfrak{R}_1\}$; $m''_1(E) = m_1(xE)$, where $E \in \mathfrak{R}''_1$, $xE \in \mathfrak{R}_1$. Then \mathfrak{R}''_1 is a ring of sets, m''_1 is a measure on \mathfrak{R}''_1 , and m''_1 and \mathfrak{R}''_1 satisfy Conditions (A) and (B).

(c) The class \mathfrak{A}_1 introduced in the proof of Theorem 5 is a translation invariant ring of subsets of G and the function μ_1 of Theorem 5 is a translation invariant measure on \mathfrak{A}_1 .

Proof. 1. (a) implies (b). The fact that \mathfrak{R}''_1 is a ring is easily verified. We now show that m''_1 is uniquely defined on \mathfrak{R}''_1 . Indeed, if $E \in \mathfrak{R}''_1$, and if both xE and yE belong to \mathfrak{R}'_1 , then xyE and yxE are in \mathfrak{R}_1 and

$$m_1(xE) = m_1(yxE) = m_1(xyE) = m_1(yE).$$

Next assume that $E_i \in \mathfrak{R}''_1$, E_i disjoint ($i = 1, 2, \dots$),

$$\bigcup_{i=1}^{\infty} E_i = E_0 \in \mathfrak{R}''_1, \quad x_i E_i \in \mathfrak{R}_1 \quad (i = 0, 1, 2, \dots).$$

Note that for each y in $x_i S$, $yE_i \in \mathfrak{R}_1$. Now (a) implies that each $x_i S$ is m'_1 -measurable and $m'_1(x_i S) = 1$. Thus, passing to \mathfrak{E}_1 and \bar{m}'_1 ,

$$\bigcap_{i=0}^{\infty} x_i S \in \mathfrak{E}_1, \quad \bar{m}'_1\left(\bigcap_{i=0}^{\infty} x_i S\right) = 1,$$

whence $\bigcap x_i S$ is not empty. Thus let $y \in \bigcap x_i S$. Then the sets yE_i are disjoint and in \mathfrak{R}_1 and

$$\bigcup_{i=1}^{\infty} yE_i = yE_0,$$

whence

$$m''_1(E_0) = m_1(yE_0) = \sum_{i=1}^{\infty} m_1(yE_i) = \sum_{i=1}^{\infty} m'_1(E_i).$$

2. (b) implies (c). Again, the fact that \mathfrak{A}_1 is a translation invariant ring is easily verified and we have already shown, in the proof of Theorem 5, that μ_1 is uniquely defined. Note that if E is such that there is some y for which $yE \in \mathfrak{R}''_1$, then $E \in \mathfrak{R}''_1$. Let μ''_1 be the function corresponding to m''_1 , and let \mathfrak{R}''_1 be the associated ring in G . By what has just been noted, \mathfrak{R}''_1 fulfils Condition (G) and thus μ''_1 is a measure on \mathfrak{R}''_1 . Since $\mathfrak{R}''_1 \supset \mathfrak{A}_1$, the assertion follows.

3. (c) implies (a). Let $\mathfrak{A}'_1 = \mathfrak{A}_1 \cup \{G - A : A \in \mathfrak{A}_1\}$. In a straightforward manner one can show that \mathfrak{A}'_1 is a ring. Condition (D) implies that $\sup\{\mu_1(gE) : g \in G, E \in \mathfrak{R}_1\} = 1$, that is, μ_1 is a bounded measure on \mathfrak{A}_1 . Let

$$\mu'_1(G - A) = 1 - \mu_1(A), \quad \mu'_1(A) = \mu_1(A), \quad A \in \mathfrak{A}_1.$$

We verify that μ'_1 is a measure on \mathfrak{A}'_1 . Let $E_i \in \mathfrak{A}'_1$, E_i disjoint and

$$\bigcup_{i=1}^{\infty} E_i = E \in \mathfrak{A}'_1.$$

We assert that if $G \notin \mathfrak{A}_1$ then at most one of the sets E_i is of the form $G - A$, $A \in \mathfrak{A}_1$. For if $(G - A) \cap (G - B)$ is empty, where A and B are in \mathfrak{A}_1 , then $G \subset (A \cup B)$ is empty, that is, $A \cup B = G$, $G \in \mathfrak{A}_1$, a contradiction. Thus either G is in \mathfrak{A}_1 or at most one of the E_i is of the form $G - A$, $A \in \mathfrak{A}_1$. In the

former case $(G, \mathfrak{H}'_1, \mu'_1) = (G, \mathfrak{H}_1, \mu_1)$. In the latter, all that need be considered is the case where E_1 , say, is of the form $G - A$, $A \in \mathfrak{H}_1$, and then

$$\sum_{i=1}^{\infty} \mu'_1(E_i) = (1 - \mu_1(A)) + \sum_{i=2}^{\infty} \mu_1(E_i).$$

Furthermore,

$$\bigcup_{i=1}^{\infty} E_i = (G - A) \cup \left(\bigcup_{i=2}^{\infty} E_i \right) = (G - A) \cup B = E,$$

and since $G - A$ and B are disjoint,

$$B = E - (G - A) = E \cap A \in \mathfrak{H}_1.$$

If $B \notin \mathfrak{H}_1$ then $B = G - C$, but then as already shown, B and $G - A$ are not disjoint, whence $B \in \mathfrak{H}_1$. Thus

$$\mu_1(B) = \sum_{i=2}^{\infty} \mu_1(E_i).$$

But $(G - A) \cup B = G - (A - B)$, and we observe $B \subset A$, whence

$$\mu'_1(G - (A - B)) = 1 - \mu_1(A - B) = 1 - (\mu_1(A) - \mu_1(B))$$

$$= 1 - \mu_1(A) + \sum_{i=2}^{\infty} \mu_1(E_i).$$

Thus μ'_1 is a measure on \mathfrak{H}'_1 .

Next, if $G - A \in \mathfrak{H}'_1$ and if $x \in G$, then

$$x(G - A) = G - xA \in \mathfrak{H}'_1,$$

$$\mu'_1(x(G - A)) = \mu'_1(G - xA) = 1 - \mu_1(xA) = 1 - \mu_1(A)$$

and so \mathfrak{H}'_1 and μ'_1 satisfy Conditions (A) and (B). Note that $(G, \mathfrak{H}'_1, \mu'_1)$ is an extension of $(G, \mathfrak{H}_1, \mu_1)$ which in turn is an extension of (G, \mathfrak{H}_1, m_1) . We assert that, in terms of the completion $\bar{\mu}'_1$ of μ'_1 , the set S is measurable. In fact, let

$$A_n \in \mathfrak{H}_1, \quad m_1(A_n) = \mu'_1(A_n) \rightarrow 1.$$

If

$$A = \bigcup_{n=1}^{\infty} A_n,$$

then A is $\bar{\mu}'_1$ -measurable and $\bar{\mu}'_1(A) = 1$, whence $\bar{\mu}'_1(G - A) = 0$. But $G - S \subset G - A$ and thus $\bar{\mu}'_1(G - S) = 0$; whence S is $\bar{\mu}'_1$ -measurable and, for $g \in G$, gS is $\bar{\mu}'_1$ -measurable and $\bar{\mu}'_1(gS) = \bar{\mu}'_1(S)$. Clearly then, the contraction of $\bar{\mu}'_1$ to the ring \mathfrak{H}'_1 is the measure m'_1 of assertion (a) of the lemma and $(S, \mathfrak{H}'_1, m'_1)$ satisfies Conditions (A) and (B).

4. (b) is true. As indicated above (part 1), we need only show that if $E_i \in \mathfrak{H}''_1$, E_i disjoint ($i = 1, 2, \dots$),

$$\bigcup_{i=1}^{\infty} E_i = E_0 \in \mathfrak{H}''_1,$$

then

$$m''_1(E_0) = \sum_{i=1}^{\infty} m''_1(E_i).$$

To this end we choose a sequence of sets $A_i \in \mathfrak{R}_1$ such that

$$m_1(A_i) > 1 - 10^{-i-1} \quad (i = 0, 1, 2, \dots).$$

Assume $x_i E_i \in \mathfrak{R}_1 (i = 0, 1, 2, \dots)$. Then

$$x_i A_i \in \mathfrak{R}_1, \quad m_1(x_i A_i) > 1 - 10^{-i-1}.$$

We consider $m_1(\bigcap_{i=0}^n x_i A_i)$. A simple induction shows

$$m_1(\bigcap_{i=0}^n x_i A_i) > 0.88 \dots 89$$

where the number of 8's preceding 9 is n . Thus, passing to \mathfrak{E}_1 and \bar{m}_1 ,

$$\bar{m}_1(\bigcap_{i=0}^{\infty} x_i A_i) > 0.8,$$

that is, $\bigcap_{i=0}^{\infty} x_i A_i$ is not empty. Let

$$y \in \bigcap_{i=0}^{\infty} x_i A_i.$$

Then $y E_i \in \mathfrak{R}_1 (i = 0, 1, 2, \dots)$, and the remainder of the proof proceeds as in part 1.

THEOREM 6. *Conditions (A B D F) imply that there is in $G = Q(S)$ a translation invariant bounded measure μ_1 . The measure induced on S considered as a subset of G coincides with the given measure. The class \mathfrak{A}_1 introduced in the proof of Theorem 5 is in fact the minimal translation invariant extension in G of \mathfrak{R}_1 .*

Proof. This is an immediate consequence of Lemma 6.

REFERENCES

1. B. R. Gelbaum, G. K. Kalisch, and J.M.H. Olmsted, *On the embedding of topological semi-groups and integral domains*, Proc. Amer. Math. Soc., vol. 2 (1951), 807-821.
2. P. R. Halmos, *Measure theory* (New York, 1950).
3. D. Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc., vol. 42 (1936), 879-882.
4. A. Weil, *L'intégration dans les groupes topologiques et ses applications* (Paris, 1938).

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A COEFFICIENT PROBLEM FOR FUNCTIONS REGULAR IN AN ANNULUS

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1. Introduction. A solution will be given in this paper for the following problem.

Let

$$(1.1) \quad w = f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be regular and single-valued in an annulus $0 < \rho < |z| < 1$. Let C_r denote the image in the w -plane of the circle $|z| = r$, $\rho < r < 1$, and let us suppose that there exists in the w -plane a straight line which cuts C_r , for every r in the range $\rho < r < 1$, in precisely $2p$ distinct points, p being a positive integer. What are the sharp bounds enforced upon the complex coefficients a_n ($n = 0, \pm 1, \dots$) by these conditions?

It will be noticed at once that, because of the presence of the arbitrary constant a_0 in (1.1), and because all the coefficients may be complex numbers, no restriction will be put upon the problem if we assume in what follows that the given straight line is the real axis.

Special cases of the above problem have been solved during the past two decades. These cases have been confined to functions $f(z)$ analytic in the circle $0 < |z| < 1$ ($\rho = 0$). If $\rho = 0$, $p = 1$, and if $f(z)$ has real coefficients in (1.1) with $a_0 = 0$, $a_1 = 1$, then $f(z)$ is typically-real for $|z| < 1$. In this case it was shown by Rogosinski [8], Dieudonné [1] and Szász [9] that

$$(1.2) \quad |a_n| \leq n|a_1|, \quad n = 2, 3, \dots$$

For $\rho = 0$, $p = 1$, $a_0 = 0$, a_n complex numbers, it was shown by the author [5] that

$$(1.3) \quad |a_n| \leq n^2|a_1|, \quad n = 2, 3, \dots$$

The inequalities expressed in (1.3) were obtained again at a later date byakeya [4]. Both (1.2) and (1.3) are sharp. When $\rho = 0$, p arbitrary, and when $a_0 = a_1 = \dots = a_{p-1} = 0$, $a_p \neq 0$, a_n complex for $n \geq p$, it was also shown [6] that

$$(1.4) \quad |a_n| \leq \frac{2}{(2p)!} \prod_{r=0}^{p-1} (n^2 - r^2) \cdot |a_p|, \quad n = p+1, p+2, \dots,$$

and (1.4) is also sharp. If, further, all the coefficients are real, the inequalities (1.4) are replaced [6] by the sharp inequalities (1.5),

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$$(1.5) \quad |a_n| < \frac{1}{n(2p-1)!} \prod_{v=0}^{n-1} (n^2 - v^2) \cdot |a_p|, \quad n = p+1, p+2, \dots$$

With the restriction that the first p coefficients be zero removed, the author showed [7] that

$$(1.6) \quad \limsup_{n \rightarrow \infty} \left| \frac{a_n}{n^{2p}} \right| < 2 \sum_{k=0}^p \frac{|a_k|}{(p+k)!(p-k)!},$$

but no sharp estimates for each individual $|a_n|$, $n > p$, were obtained. Quite recently Goodman and Robertson [3] have solved the problem for $\rho = 0$, p arbitrary, when the coefficients are all real, obtaining the sharp inequalities for $n > p$,

$$(1.7) \quad |a_n| < \sum_{k=1}^p \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|,$$

which is of particular interest since it establishes, for a large class of multivalent functions, the Goodman conjecture [2] that (1.7) holds for all functions multivalent of order p in $|z| < 1$.

In this paper we turn to the general case. Thus p is an arbitrary positive integer. The coefficients a_n ($n = 0, \pm 1, \pm 2, \dots$) are complex numbers and ρ may be positive or zero. We shall denote by \bar{a} the complex conjugate of the number a . Our main result is stated in the following theorem.

THEOREM 1. *Let*

$$(1.8) \quad w = f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be regular and single-valued for $0 < \rho < |z| < 1$, the coefficients a_n in (1.8) being complex numbers. On each circle $|z| = r$, $\rho < r < 1$, let the imaginary part of $f(z)$ change sign $2p$ times, where p is a positive integer independent of r . Then, for $n > p$, the following inequalities hold and are sharp in all the variables $|a_k - \bar{a}_{-k}|$ ($k = 0, 1, \dots, p$):

$$(1.9) \quad |a_n - \bar{a}_{-n}| < \sum_{k=0}^p \Delta(p, k, n) |a_k - \bar{a}_{-k}|,$$

where

$$\Delta(p, 0, n) \equiv \prod_{v=1}^p (n^2 - v^2) / (p!)^2,$$

$$\Delta(p, k, n) \equiv 2 \prod_{\substack{v=0 \\ v \neq k}}^p (n^2 - v^2) / (p+k)!(p-k)!, \quad p > k > 0.$$

By a rotation and translation we may obtain the following Corollary to Theorem 1.

COROLLARY. *Let $f(z)$ be given as in (1.8), regular and single-valued for $\rho < |z| < 1$. Let l be any straight line in the w -plane. Denote by $de^{i\gamma}$ ($d > 0$), the point on l nearest the origin. For each r in the range $\rho < r < 1$ let the image*

curve C , of $|z| = r$, through the mapping $w = f(z)$, cross l precisely $2p$ times (p fixed). Then for $n > p$,

$$(1.10) \quad |a_n e^{-i\gamma} + \bar{a}_n e^{i\gamma}| \leq 2\Delta(p, 0, n) |\Re(a_0 e^{i\gamma}) - d| \\ + \sum_{k=1}^p \Delta(p, k, n) |a_k e^{-i\gamma} + \bar{a}_k e^{i\gamma}|.$$

In the corollary, if $d = 0$ we define γ to be $\frac{1}{2}\pi + \beta$, $0 < \beta < \pi$, where β is the inclination of the line l . If l is the real axis, then $d = \beta = 0$, $\gamma = \frac{1}{2}\pi$, and (1.10) reduces to (1.9).

We call attention to the fact that for $0 < k \leq p$,

$$(1.11) \quad \Delta(p, k, n) = \frac{n}{k} D(p, k, n)$$

where $D(p, k, n)$ is the coefficient of $|a_k|$ in (1.7).

The special case of Theorem 1 where $p = 1$ states that

$$(1.12) \quad |a_n - \bar{a}_{-n}| \leq |n^2 - 1| \cdot |a_0 - \bar{a}_0| + n^2 |a_1 - \bar{a}_{-1}|$$

for all integers n . (1.12) includes (1.3) as a special case. It is also seen that (1.6) follows as another special case of (1.9).

From the identity

$$(1.13) \quad \Im\{zf'(z)\} = -\frac{\partial}{\partial \theta} \Re\{f(re^{i\theta})\}, \quad z = re^{i\theta},$$

the following theorem is obtained immediately from Theorem 1. It shows that the Goodman conjecture (1.7) for all multivalent functions of order p in $|z| < 1$ holds for still another special class of multivalent functions of order p , this time with the coefficients complex numbers. We state the result as

THEOREM 2. Let

$$(1.14) \quad f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

have complex coefficients and be regular and multivalent of order p for $|z| < 1$, and for each r of the interval $\rho < r < 1$ let

$$\frac{\partial}{\partial \theta} \Re\{f(re^{i\theta})\}$$

change sign $2p$ times on $|z| = r$. Then for $n > p$,

$$(1.15) \quad |a_n| \leq \sum_{k=1}^p \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|,$$

and this bound is sharp in all the variables $|a_k|$ ($k = 1, 2, \dots, p$).

Theorem 2 was shown [5] in the special case $p = 1$, in which case $w = f(z)$ is univalent (*schlicht*) for $|z| < 1$ and maps each circle $|z| = r$, $\rho < r < 1$, onto a contour with the property that each line parallel to the imaginary axis cuts the contour in at most two points. In Theorem 2 with $p \geq 1$, each such line is cut in at most $2p$ points.

2. Preliminary remarks. For the sake of clarity we give here a few definitions from [3].

Definition 1. The harmonic function $v(r, \theta)$ is said to have a *change of sign* at $\theta = \theta_j$ if there exists an $\epsilon > 0$ such that, for $0 < \delta < \epsilon$,

$$(2.1) \quad v(r, \theta_j + \delta) \cdot v(r, \theta_j - \delta) < 0.$$

Definition 2. $\Im f(z) = v(r, \theta)$ is said to *change sign q times* on $|z| = r$ if there are q values of θ , say $\theta_1, \theta_2, \dots, \theta_q$ such that

- (a) inequality (2.1) holds for each θ_j ($j = 1, 2, \dots, q$),
- (b) $\theta_j \not\equiv \theta_k \pmod{2\pi}$ if $j \neq k$,
- (c) if θ_j is any value of θ for which $v(r, \theta)$ has a change of sign then, for one of the θ_j ($j = 1, \dots, q$), $\theta = \theta_j \pmod{2\pi}$.

In proving (1.9) of Theorem 1 we may assume that $f(z)$, given by (1.8), is regular on $|z| = 1$. For if this is not the case and $f(z)$ is regular for $0 < \rho < |z| < 1$ and $f(z) = f_-(z) + f_+(z)$, where

$$f_+(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_-(z) = \sum_{n=1}^{\infty} a_{-n} z^{-n},$$

then, for t sufficiently near to, but less than, one,

$$(2.2) \quad f_t(z) = f_+(tz) + f_-(z/t)$$

is regular on $|z| = 1$. Having proved (1.9) for $f_t(z)$ we have only to let $t \rightarrow 1$ to obtain (1.9) for $f(z)$. Hence in what follows we shall assume $f(z)$ to be regular on $|z| = 1$.

LEMMA 1. Let $f(z)$ be given by (1.8) and be regular for $\rho < |z| \leq 1$. Let $\Im f(e^{i\theta})$ change sign $2p$ times at $\theta = \theta_1, \theta_2, \dots, \theta_{2p}$ ($p \geq 1$). There exist real numbers μ and ν such that, if $g(z)$ is defined by

$$(2.3) \quad g(z) = (z + z^{-1} - 2 \cos \nu) f(z e^{i\nu}) = \sum_{n=-\infty}^{\infty} b_n z^n,$$

then $\Im g(e^{i\theta})$ changes sign $2(p-1)$ times.

Proof. Since $\Im g(e^{i\theta}) = 2(\cos \theta - \cos \nu) \Im f(e^{i(\theta+\nu)})$ we may take

$$(2.4) \quad 2\mu = \theta_i + \theta_j, \quad 2\nu = \theta_j - \theta_i,$$

where θ_i and θ_j , $i \neq j$, are any two values of θ where $\Im f(e^{i\theta})$ changes sign. Then $\Im f(e^{i(\theta+\nu)})$ changes sign at $\theta = \pm(\theta_j - \theta_i)/2 = \pm \nu$ as well as at $2(p-1)$ other values of θ . However, $(\cos \theta - \cos \nu)$ changes sign exactly twice in $-\pi < \theta < \pi$, at $\theta = \pm \nu$. Thus $\Im g(e^{i\theta})$ does not change sign at $\pm \nu$ but does change sign at the $2(p-1)$ other values of θ (nowhere else if $p = 1$).

The method of proof for (1.9) will be by induction on p by means of (2.3). Note that even in the special case where $f(z)$ is regular in the whole unit circle $|z| < 1$, $g(z)$ is not regular at $z = 0$ when $a_0 \neq 0$. This difficulty was easily avoided [3] when all the coefficients were real and $\mu = 0$.

The proof of (1.7) by induction on p made use of the fact that (1.7) was already known for $p = 1$, i.e., that (1.2) held. However, (1.9) is not known to be true even for $p = 1$, except in the special case (1.3). It will therefore be necessary to establish (1.12) first. In proving (1.12) we shall need the following lemma which appears in [7, p. 514].

LEMMA 2. Let

$$(2.5) \quad F(z) = \sum_{n=0}^{\infty} A_n z^n$$

be analytic and single-valued for $\rho < |z| < 1$. If $\Re F(re^{i\theta}) > 0$ for $\rho < r < 1$, then

$$(2.6) \quad |A_n + \bar{A}_{-n}| < 2\Re A_0,$$

where \bar{A}_{-n} denotes the complex conjugate of A_{-n} . If $F(z)$ is regular on $|z| = 1$, and if $\Re F(e^{i\theta}) > 0$, then (2.6) again holds.

3. Proof of (1.12). Let

$$(3.1) \quad \phi(z) = c_0 + c_1 z + \dots + c_n z^n + \dots$$

be regular for $|z| < 1$ and let $\Im \phi(z)$ change sign exactly twice on $|z| = 1$. By Lemma 1, μ and ν exist so that

$$(3.2) \quad \begin{aligned} g(z) &= (z + z^{-1} - 2 \cos \nu) \phi(z e^{i\mu}) \\ &= b_{-1} z^{-1} + b_0 + b_1 z + \dots + b_n z^n + \dots \end{aligned}$$

is regular on $|z| = 1$ and $\Im g(z)$ does not change sign on $|z| = 1$. Since $ig(z)$ (or $-ig(z)$) satisfies the conditions of Lemma 2, we have

$$(3.3) \quad |b_1 - \bar{b}_{-1}| < |\Im(b_0 - \bar{b}_0)|,$$

$$(3.4) \quad |b_n| < |\Im(b_0 - \bar{b}_0)|, \quad n > 1.$$

From (3.1) and (3.2) a comparison of coefficients in the two power series gives

$$(3.5) \quad b_{-1} = c_0, \quad b_0 = c_1 e^{i\mu} - 2c_0 \cos \nu,$$

$$(3.6) \quad b_n = c_{n+1} e^{(n+1)i\mu} - 2c_n e^{ni\mu} \cos \nu + c_{n-1} e^{(n-1)i\mu}, \quad n \geq 1,$$

$$(3.7) \quad c_n e^{ni\mu} = \sum_{k=1}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu}.$$

Substituting for b_{-1} and b_0 from (3.5) in (3.7) we have

$$(3.8) \quad \begin{aligned} c_n e^{ni\mu} &= c_0 \frac{\sin(n+1)\nu}{\sin \nu} + (c_1 e^{i\mu} - 2c_0 \cos \nu) \frac{\sin n\nu}{\sin \nu} + \sum_{k=1}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu} \\ &= c_0 \left\{ \frac{\sin(n+1)\nu}{\sin \nu} - 2 \cos \nu \frac{\sin n\nu}{\sin \nu} \right\} + c_1 e^{i\mu} \frac{\sin n\nu}{\sin \nu} + \sum_{k=1}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu} \\ &= -c_0 \frac{\sin(n-1)\nu}{\sin \nu} + c_1 e^{i\mu} \frac{\sin n\nu}{\sin \nu} + \sum_{k=1}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu}. \end{aligned}$$

From (3.8) we have the inequalities

$$(3.9) \quad |c_n| < (n-1)|c_0| + n|c_1| + \sum_{k=2}^{n-1} (n-k)|b_k|, \quad n > 1.$$

From (3.3), (3.4), and (3.5) we have

$$(3.10) \quad |b_1| < |b_{-1}| + |\Im(b_0 - \bar{b}_0)| < 5|c_0| + 2|c_1|,$$

$$(3.11) \quad |b_k| < 2|b_0| < 4|c_0| + 2|c_1|, \quad k > 1.$$

Substituting (3.10) and (3.11) in (3.9) we obtain

$$\begin{aligned} |c_n| &< 6(n-1)|c_0| + (3n-2)|c_1| + \sum_{k=2}^{n-1} (n-k)(4|c_0| + 2|c_1|) \\ &= |c_0| \left\{ 6n - 6 + 4 \sum_{k=2}^{n-1} (n-k) \right\} + |c_1| \left\{ 3n - 2 + 2 \sum_{k=2}^{n-1} (n-k) \right\}. \end{aligned}$$

Thus for all integers n ,

$$(3.12) \quad |c_n| < 2|n^2 - 1| \cdot |c_0| + n^2|c_1|,$$

and we now have proved a special case of (1.12).

Let now

$$(3.13) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be regular on $|z| = 1$ and such that $\Im f(z)$ changes sign exactly twice on $|z| = 1$. We may write $f(z)$ as

$$(3.14) \quad f(z) = \sum_1^{\infty} (a_{-n} z^{-n} + \bar{a}_{-n} z^n) + a_0 + \sum_{n=1}^{\infty} (a_n - \bar{a}_n) z^n$$

where both series in (3.14) converge on $|z| = 1$. On $|z| = 1$,

$$(3.15) \quad \Im \left\{ \sum_{n=1}^{\infty} (a_{-n} z^{-n} + \bar{a}_{-n} z^n) + \Re a_0 \right\} \equiv 0.$$

In this case, from (3.14) and (3.15) it follows that on $|z| = 1$ the imaginary part of the function

$$(3.16) \quad \phi^*(z) = (\Im a_0)i + \sum_{n=1}^{\infty} (a_n - \bar{a}_n) z^n$$

changes sign exactly twice. Thus $\phi^*(z)$ behaves like $\phi(z)$ in (3.1) and inequalities corresponding to (3.12) must hold for $\phi^*(z)$. This then gives for all integers n ,

$$(3.17) \quad |a_n - \bar{a}_{-n}| < |n^2 - 1| \cdot |a_0 - \bar{a}_0| + n^2 |a_1 - \bar{a}_{-1}|.$$

We note that in (3.17) not both the terms $|a_0 - \bar{a}_0|$ and $|a_1 - \bar{a}_{-1}|$ can vanish simultaneously. For otherwise the function

$$\phi^*(z) = (a_2 - \bar{a}_{-2})z^2 + \dots + (a_n - \bar{a}_{-n})z^n + \dots,$$

regular for $|z| \leq 1$, would map $|z| = r$, for every r in $0 < r < 1$, into a contour which cuts the real axis at least four times. This, however, is contrary to the hypothesis for $\phi^*(z)$ since $p = 1$. The proof of (1.12) is now complete.

4. Proof of Theorem 1. We now let

$$(4.1) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be regular and single-valued for $0 < \rho < |z| < 1$. We suppose that $\Im f(z)$ changes sign exactly $2p$ times on $|z| = 1$. By Lemma 1 there exist real numbers μ and ν such that

$$(4.2) \quad g(z) = (z + z^{-1} - 2 \cos \nu) f(z e^{i\nu}) = \sum_{n=-\infty}^{\infty} b_n z^n$$

is regular and single-valued for $\rho < |z| < 1$ and $\Im g(z)$ changes sign $2(p-1)$ times on $|z| = 1$.

A comparison of coefficients in (4.1) and (4.2) gives

$$(4.3) \quad b_n = a_{n+1} e^{(n+1)\mu i} - 2a_n e^{n\mu i} \cos \nu + a_{n-1} e^{(n-1)\mu i}, \quad n = 0, \pm 1, \pm 2, \dots,$$

$$(4.4) \quad a_n e^{n\mu i} = \sum_{k=-\infty}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu} = S_1 + S_2,$$

where

$$(4.5) \quad S_1 = \sum_{k=-\infty}^{p-1} b_k \frac{\sin(n-k)\nu}{\sin \nu},$$

$$(4.6) \quad S_2 = \sum_{k=p}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu}.$$

Making use of (4.3) in (4.5) we simplify S_1 as follows:

$$\begin{aligned} S_1 &= \sum_{k=-\infty}^{p-1} (a_{k+1} e^{(k+1)\mu i} - 2a_k e^{k\mu i} \cos \nu + a_{k-1} e^{(k-1)\mu i}) \frac{\sin(n-k)\nu}{\sin \nu} \\ &= a_p e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} + a_{p-1} e^{(p-1)\mu i} \left\{ \frac{\sin(n-p+2)\nu}{\sin \nu} \right. \\ &\quad \left. - 2 \cos \nu \frac{\sin(n-p+1)\nu}{\sin \nu} \right\} \\ &\quad + \sum_{k=-\infty}^{p-3} a_k e^{k\mu i} \left\{ \frac{\sin(n-k+1)\nu}{\sin \nu} - 2 \cos \nu \frac{\sin(n-k)\nu}{\sin \nu} + \frac{\sin(n-k-1)\nu}{\sin \nu} \right\}, \end{aligned}$$

so that

$$(4.7) \quad S_1 \equiv a_p e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} - a_{p-1} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu}.$$

Thus (4.4) simplifies to

$$(4.8) \quad a_n e^{n\mu i} = a_p e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} - a_{p-1} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} + \sum_{k=p}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin \nu}.$$

Next, consider the expression

$$\begin{aligned}
 (4.9) \quad T &= \sum_{k=p}^{n-1} \frac{\delta_{-k} \sin(n-k)\nu}{\sin \nu} \\
 &= \sum_{k=p}^{n-1} \{ \bar{a}_{1-k} e^{(k-1)\mu i} - 2\bar{a}_{-k} e^{k\mu i} \cos \nu + \bar{a}_{-k-1} e^{(k+1)\mu i} \} \frac{\sin(n-k)\nu}{\sin \nu} \\
 &= \bar{a}_{1-p} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} \\
 &\quad + \bar{a}_{-p} e^{p\mu i} \left\{ \frac{\sin(n-p-1)\nu}{\sin \nu} - 2 \cos \nu \frac{\sin(n-p)\nu}{\sin \nu} \right\} + \bar{a}_{-n} e^{n\mu i} \\
 &\quad + \sum_{k=p+1}^{n-1} \bar{a}_{-k} e^{k\mu i} \left\{ \frac{\sin(n-k-1)\nu}{\sin \nu} - 2 \cos \nu \frac{\sin(n-k)\nu}{\sin \nu} + \frac{\sin(n-k+1)\nu}{\sin \nu} \right\} \\
 (4.10) \quad T &= \bar{a}_{1-p} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} - \bar{a}_{-p} e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} + \bar{a}_{-n} e^{n\mu i}.
 \end{aligned}$$

Adding to the right-hand side of (4.8) the expression for T in (4.10) and then subtracting the expression for T in (4.9) we may rewrite (4.8) in the form

$$\begin{aligned}
 (4.11) \quad a_n e^{n\mu i} &= \left\{ \bar{a}_{1-p} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} - \bar{a}_{-p} e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} + \bar{a}_{-n} e^{n\mu i} \right\} \\
 &\quad + \left\{ \bar{a}_{-p} e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} - \bar{a}_{p-1} e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} \right\} \\
 &\quad + \sum_{k=p}^{n-1} (b_k - \bar{b}_{-k}) \frac{\sin(n-k)\nu}{\sin \nu},
 \end{aligned}$$

$$\begin{aligned}
 (4.12) \quad (a_n - \bar{a}_{-n}) e^{n\mu i} &= (a_p - \bar{a}_{-p}) e^{p\mu i} \frac{\sin(n-p+1)\nu}{\sin \nu} \\
 &\quad - (a_{p-1} - \bar{a}_{1-p}) e^{(p-1)\mu i} \frac{\sin(n-p)\nu}{\sin \nu} \\
 &\quad + \sum_{k=p}^{n-1} (b_k - \bar{b}_{-k}) \frac{\sin(n-k)\nu}{\sin \nu},
 \end{aligned}$$

$$\begin{aligned}
 (4.13) \quad |a_n - \bar{a}_{-n}| &< (n-p+1)|a_p - \bar{a}_{-p}| + (n-p)|a_{p-1} - \bar{a}_{1-p}| \\
 &\quad + \sum_{k=p}^{n-1} (n-k)|b_k - \bar{b}_{-k}|.
 \end{aligned}$$

We have already seen that (1.9) holds for $p=1$ and all integers $n>1$ because of (3.17). We now assume that the coefficients b_n of $g(z)$ in (4.2) satisfy (1.9) with $p-1$ replacing p and b_n replacing a_n . Then from (4.13) we shall be able to prove by induction that (1.9) holds for all p and all $n>p$. Thus we assume, for $k>p-1$,

$$(4.14) \quad |b_k - \bar{b}_{-k}| < \sum_{s=0}^{p-1} \Delta(p-1, s, k) |b_s - \bar{b}_{-s}|.$$

From (4.3) we have the inequalities

$$(4.15) \quad |b_0 - \bar{b}_0| \leq 2|a_0 - \bar{a}_0| + 2|a_1 - \bar{a}_1|,$$

$$(4.16) \quad |b_s - \bar{b}_s| \leq |a_{s+1} - \bar{a}_{s-1}| + 2|a_s - \bar{a}_s| + |a_{s-1} - \bar{a}_{s-1}|, \\ s = 1, 2, \dots, p-1.$$

Making use of (4.15) and (4.16) in (4.14) we have

$$(4.17) \quad |b_k - \bar{b}_k| \leq 2\Delta(p-1, 0, k)(|a_0 - \bar{a}_0| + |a_1 - \bar{a}_1|) + R,$$

where

$$(4.18) \quad R = \sum_{s=1}^{p-1} \Delta(p-1, s, k)(|a_{s+1} - \bar{a}_{s-1}| + 2|a_s - \bar{a}_s| + |a_{s-1} - \bar{a}_{s-1}|).$$

Collecting terms in (4.17) and (4.18), we may write

$$(4.19) \quad |b_k - \bar{b}_k| \leq \frac{2k(k+p-1)!}{(k-p)!} \left[\frac{1}{((p-1)!)^2 k^2} + \frac{1}{p!(p-2)!(k^2-1)} \right] |a_0 - \bar{a}_0| \\ + \frac{2k(k+p-1)!}{(k-p)!} \left[\frac{1}{(2p-2)!(k^2-(p-1)^2)} \right] |a_p - \bar{a}_p| \\ + \frac{2k(k+p-1)!}{(k-p)!} \left[\frac{1}{(2p-3)!(k^2-(p-2)^2)} \right. \\ \left. + \frac{2}{(2p-2)!(k^2-(p-1)^2)} \right] |a_{p-1} - \bar{a}_{p-1}| \\ + \sum_{s=1}^{p-1} \frac{2k(k+p-1)!}{(k-p)!} \left[\frac{|a_s - \bar{a}_s|}{(p-2+s)!(p-s)!(k^2-(s-1)^2)} \right. \\ \left. + \frac{2|a_s - \bar{a}_s|}{(p-1+s)!(p-1-s)!(k^2-s^2)} \right. \\ \left. + \frac{|a_s - \bar{a}_s|}{(p+s)!(p-2-s)!(k^2-(s+1)^2)} \right].$$

Substituting (4.19) in (4.13), we obtain

$$(4.20) \quad |a_n - \bar{a}_n| \leq \sum_{p=0}^p D_p |a_p - \bar{a}_p|,$$

where

$$(4.21) \quad D_0 = \sum_{k=p}^{n-1} \frac{2k(n-k)(k+p-1)!}{(k-p)!} \left[\frac{2}{((p-1)!)^2 k^2} + \frac{1}{p!(p-2)!(k^2-1)} \right],$$

$$(4.22) \quad D_{p-1} = \frac{4}{(2p-2)!} \sum_{k=p-2}^{n-1} \frac{k(n-k)(k+p-2)!}{(k+p-2)(k-p+2)!} \\ \cdot [(p-1)(k^2-(p-1)^2) + k^2 - (p-2)^2],$$

$$(4.23) \quad D_p = \frac{2}{(2p-2)!} \sum_{k=p-1}^{n-1} \frac{k(n-k)(k+p-2)!}{(k-p+1)!},$$

while for $\mu = 1, 2, \dots, p-2$ we have

$$(4.24) \quad D_\mu = \sum_{k=p}^{n-1} \frac{2k(n-k)(k+p-1)!}{(k-p)!} A_k$$

where

$$A_k = \left[\frac{1}{(\mu-2+\mu)!(\mu-\mu)!(k^2 - (\mu-1)^2)} + \frac{2}{(\mu-1+\mu)!(\mu-1-\mu)!(k^2 - \mu^2)} + \frac{1}{(\mu+\mu)!(\mu-2-\mu)!(k^2 - (\mu+1)^2)} \right].$$

To complete the proof of (1.9) by induction it is then sufficient to evaluate D_μ ($\mu = 0, 1, \dots, p$) and, indeed, to show that

$$(4.25) \quad D_\mu = \Delta(p, \mu, n).$$

5. Formulae for D_μ . We shall prove first (4.25) in the case $\mu = p$, that is,

$$(5.1) \quad \frac{2}{(2p-2)!} \sum_{k=p-1}^{n-1} \frac{k(n-k)(k+p-2)!}{(k-p+1)!} = \frac{2n(n+p)!}{(2p)!(n-p-1)!(n^2 - p^2)}.$$

Equation (5.1) is equivalent to

$$(5.2) \quad \sum_{k=p-1}^{n-1} \frac{k(n-k)(k+p-2)!}{(k-p+1)!} = \frac{n(n+p-1)!}{2p(2p-1)(n-p)!}$$

which is easily seen to be true when $n = p$. We shall prove (5.2) by induction on n , making use of the formula

$$(5.3) \quad \sum_{s=0}^n \frac{(s+q)!}{s!} = \frac{(n+q+1)!}{(q+1) \cdot (n)!}.$$

Assuming (5.2) for an integer n , we then have

$$\begin{aligned} (5.4) \quad \sum_{k=p-1}^n \frac{k(n+1-k)(k+p-2)!}{(k-p+1)!} &= \frac{n(n+p-2)!}{(n-p+1)!} + \sum_{k=p-1}^{n-1} \frac{k(n+1-k)(k+p-2)!}{(k-p+1)!} \\ &= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \sum_{k=p-1}^{n-1} \frac{k(k+p-2)!}{(k-p+1)!} \\ &= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \sum_{k=p-1}^{n-1} \frac{(k+p-1)!}{(k-p+1)!} \\ &\quad - (p-1) \sum_{k=p-1}^{n-1} \frac{(k+p-2)!}{(k-p+1)!} \end{aligned}$$

$$\begin{aligned}
&= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \frac{(n+p-1)!}{(2p-1)(n-p)!} \\
&\quad - \frac{(p-1)(n+p-2)!}{(2p-2)(n-p)!} \\
&= \frac{(n+1)(n+p)!}{2p(2p-1)(n+1-p)!}
\end{aligned}$$

Thus (5.2)_n holds when n is replaced by $n+1$. This completes the proof by induction of (5.1).

Next we shall establish (4.25) for the values $\mu = 1, 2, \dots, p-2$. We shall omit the proof of (4.25) in the cases $\mu = 0, \mu = p-1$ since the method of proof in these two cases is very similar to the typical case which follows. We assume now that $1 < \mu < p-2, n > p$, and shall prove that

$$\begin{aligned}
(5.6) \quad \sum_{k=p}^{n-1} \frac{k(n-k)(k+p-1)!}{(k-p)!} &\left[\frac{(p+\mu-1)(p+\mu)}{k^3 - (\mu-1)^3} + \frac{2(p^2 - \mu^2)}{k^3 - \mu^3} \right. \\
&\quad \left. + \frac{(p-\mu-1)(p-\mu)}{k^3 - (\mu+1)^3} \right] \\
&= \frac{n(n+p)!}{(n-p-1)!(n^3 - \mu^3)},
\end{aligned}$$

which is equivalent to (4.25). It can be easily verified that (5.6) holds when $n = p+1$. We assume (5.6) for an integer n and prove that (5.6) holds for the integer $n+1$. Replacing n by $n+1$ in the left-hand side of equation (5.6) we have

$$\begin{aligned}
(5.7) \quad \sum_{k=p}^n \frac{k(n+1-k)(k+p-1)!}{(k-p)!} &\left[\frac{(p+\mu)^2 - (p+\mu)}{k^3 - (\mu-1)^3} + \frac{2(p^2 - \mu^2)}{k^3 - \mu^3} \right. \\
&\quad \left. + \frac{(p-\mu)^2 - (p-\mu)}{k^3 - (\mu+1)^3} \right] = \sum_{k=p}^n + \sum_{k=p}^{n-1} \\
&= \frac{n(n+p-1)!}{(n-p)!} \left[\frac{(p+\mu)^2 - (p+\mu)}{n^3 - (\mu-1)^3} + \frac{2(p^2 - \mu^2)}{n^3 - \mu^3} + \frac{(p-\mu)^2 - (p-\mu)}{n^3 - (\mu+1)^3} \right] \\
&\quad + \frac{n(n+p)!}{(n-p-1)!(n^3 - \mu^3)} + \sum_{k=p}^{n-1} \frac{k(k+p-1)!}{(k-p)!} \left[\frac{(p+\mu)^2 - (p+\mu)}{k^3 - (\mu-1)^3} \right. \\
&\quad \left. + \frac{2(p^2 - \mu^2)}{k^3 - \mu^3} + \frac{(p-\mu)^2 - (p-\mu)}{k^3 - (\mu+1)^3} \right].
\end{aligned}$$

The right-hand side of (5.7) should equal

$$\frac{(n+1)(n+p+1)!}{(n-p)!(n+1)^3 - \mu^3}$$

if (5.6) is to be proven, and we can see that it does so, provided we show that

$$\begin{aligned}
 (5.8) \quad & \sum_{k=p}^{n-1} \frac{k(k+p-1)!}{(k-p)!} \left[\frac{(p+\mu)^2 - (p+\mu)}{k^2 - (\mu-1)^2} \right. \\
 & \quad \left. + \frac{2(p^2 - \mu^2)}{k^2 - \mu^2} + \frac{(p-\mu)^2 - (p-\mu)}{k^2 - (\mu+1)^2} \right] \\
 & = \frac{(n+p-1)!}{(n-p)!} \left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2 - \mu^2} - \frac{n(n^2 - p^2)}{n^2 - \mu^2} \right. \\
 & \quad \left. - n \left\{ \frac{(p+\mu)^2 - (p+\mu)}{n^2 - (\mu-1)^2} + \frac{2(p^2 - \mu^2)}{n^2 - \mu^2} + \frac{(p-\mu)^2 - (p-\mu)}{n^2 - (\mu+1)^2} \right\} \right].
 \end{aligned}$$

We shall prove (5.8) by induction on n . It is easily verified that (5.8) holds for $n = p + 1$. Replacing n by $n + 1$ in the left-hand side of (5.8), we have

$$\begin{aligned}
 (5.9) \quad & \sum_{k=p}^n \frac{k(k+p-1)!}{(k-p)!} \left[\frac{(p+\mu)^2 - (p+\mu)}{k^2 - (\mu-1)^2} + \frac{2(p^2 - \mu^2)}{k^2 - \mu^2} \right. \\
 & \quad \left. + \frac{(p-\mu)^2 - (p-\mu)}{k^2 - (\mu+1)^2} \right] = \sum_{k=p}^n + \sum_{k=p}^{n-1} \\
 & = \frac{n(n+p-1)!}{(n-p)!} \left[\frac{(p+\mu)^2 - (p+\mu)}{n^2 - (\mu-1)^2} + \frac{2(p^2 - \mu^2)}{n^2 - \mu^2} + \frac{(p-\mu)^2 - (p-\mu)}{n^2 - (\mu+1)^2} \right] \\
 & \quad + \frac{(n+p-1)!}{(n-p)!} \left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2 - \mu^2} - \frac{n(n^2 - p^2)}{n^2 - \mu^2} \right. \\
 & \quad \left. - n \left\{ \frac{(p+\mu)^2 - (p+\mu)}{n^2 - (\mu-1)^2} + \frac{2(p^2 - \mu^2)}{n^2 - \mu^2} + \frac{(p-\mu)^2 - (p-\mu)}{n^2 - (\mu+1)^2} \right\} \right] \\
 & = \frac{(n+p-1)!}{(n-p)!} \left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2 - \mu^2} - \frac{n(n^2 - p^2)}{n^2 - \mu^2} \right] \\
 & = \frac{(n+p)!}{(n-p)!} \left[\frac{(n+1)(n+p+1)}{(n+1)^2 - \mu^2} - \frac{n(n-p)}{n^2 - \mu^2} \right].
 \end{aligned}$$

If (5.8) is to be proven the right-hand side of (5.9) should be the same as the expression obtained by replacing n by $n + 1$ in the right-hand side of (5.8). This will be the case, provided we prove that

$$\begin{aligned}
 (5.10) \quad & (n+1-p) \left[\frac{(n+1)(n+p+1)}{(n+1)^2 - \mu^2} - \frac{n(n-p)}{n^2 - \mu^2} \right] \\
 & = \left[\frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^2 - \mu^2} - \frac{(n+1)((n+1)^2 - p^2)}{(n+1)^2 - \mu^2} \right. \\
 & \quad \left. - (n+1) \left\{ \frac{(p+\mu)^2 - (p+\mu)}{(n+1)^2 - (\mu-1)^2} + \frac{2(p^2 - \mu^2)}{(n+1)^2 - \mu^2} \right. \right. \\
 & \quad \left. \left. + \frac{(p-\mu)^2 - (p-\mu)}{(n+1)^2 - (\mu+1)^2} \right\} \right].
 \end{aligned}$$

(5.10) is equivalent to

$$\begin{aligned}
 (5.11) \quad & \frac{2(n+1)^3 - 2p^2(n+1)}{(n+1)^3 - \mu^3} - \frac{n(n-p)(n-p+1)}{n^3 - \mu^3} \\
 &= \frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^3 - \mu^3} \\
 &\quad - (n+1) \left\{ \frac{(p+\mu)^2 - (p+\mu)}{(n+1)^3 - (\mu-1)^3} + \frac{2(p^2 - \mu^2)}{(n+1)^3 - \mu^3} \right. \\
 &\quad \left. + \frac{(p-\mu)^2 - (p-\mu)}{(n+1)^3 - (\mu+1)^3} \right\}.
 \end{aligned}$$

If all terms with denominator $(n+1)^3 - \mu^3$ in (5.11) are collected and placed on the left-hand side, these terms simplify to the simple term $(2n+2)$ so that (5.11) is equivalent to

$$\begin{aligned}
 (5.12) \quad 2n+2 &= \frac{n(n-p)(n-p+1)}{n^3 - \mu^3} + \frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^3 - \mu^3} \\
 &\quad - \frac{(n+1)(p+\mu)^2 - (n+1)(p+\mu)}{(n+\mu)(n+2-\mu)} \\
 &\quad - \frac{(n+1)(p-\mu)^2 - (n+1)(p-\mu)}{(n-\mu)(n+2+\mu)},
 \end{aligned}$$

and (5.12) is equivalent to

$$\begin{aligned}
 (5.13) \quad & (2n+2)(n^3 - \mu^2)(n^3 + 4n + 4 - \mu^3) \\
 &= n(n-p)(n-p+1)(n^3 + 4n + 4 - \mu^3) \\
 &\quad + (n+2)(n+p+2)(n+p+1)(n^3 - \mu^3) \\
 &\quad - (n+1)(n-\mu)(n+\mu+2)(p+\mu)^2 \\
 &\quad + (n+1)(n-\mu)(n+\mu+2)(p+\mu) \\
 &\quad - (n+1)(n+\mu)(n-\mu+2)(p-\mu)^2 \\
 &\quad + (n+1)(n+\mu)(n-\mu+2)(p-\mu).
 \end{aligned}$$

Both sides of (5.13) reduce to the polynomial

$$2n^5 + 10n^4 + (16 - 4\mu^2)n^3 + (8 - 12\mu^2)n^2 + (2\mu^4 - 16\mu^2)n + (2\mu^4 - 8\mu^3).$$

Since (5.13) is therefore an identity, this completes the proof by induction of (5.6). Thus (4.25) is established.

6. Sharpness of Theorems 1 and 2. We shall show now that inequalities (1.9), (1.10), and (1.15) are sharp for all integers p . Since the quantities

$$|a_k - \bar{a}_k|, \quad k = 0, 1, \dots, p,$$

are to be assigned arbitrary values in advance, in order to prove that (1.9) is

sharp it will be sufficient to exhibit a function $w = f^*(z)$ of the form (1.8) which is a power series. Thus we shall take $a_{-k} = 0$ for all $k > 0$. Since the addition of a real constant to the function $f^*(z)$ does not affect its imaginary part, we may assume that a_0 is a pure imaginary number id , $d > 0$, without restricting the problem. Then, having shown that (1.9) is sharp by exhibiting a function $f^*(z)$ satisfying the conditions of Theorem 1, for which equality signs hold for all $n > p$ in (1.9), we see at once by a rotation and translation that (1.10) is also sharp; and from (1.13) that (1.15) is indeed sharp too.

Let $|a_0|, |a_1|, \dots, |a_p|$ be p arbitrary non-negative numbers, not all zero. Define

$$(6.1) \quad a_k = (-1)^{k,k+1} |a_k|, \quad k = 0, 1, \dots, p,$$

$$(6.2) \quad A_k = (2k)! \sum_{q=0}^k \frac{|a_q|}{(k+q)!(k-q)!}, \quad k = 0, 1, \dots, p,$$

$$(6.3) \quad f^*(z) = \sum_{k=0}^p (-1)^{k,k+1} A_k \frac{z^k + iz^{k+1}}{(1-iz)^{2k+1}} = \sum_{k=0}^{\infty} a_k z^k.$$

We shall show that $f^*(z)$ satisfies the conditions of Theorem 1, and that equality signs hold in (1.9) for this function. We must show that there exists an interval $\rho < r < 1$ such that on each circle $|z| = r$ of this interval the imaginary part of $f^*(z)$ changes sign exactly $2p$ times. To this end it will be sufficient to consider the real part of the function $-if^*(-iz)$ for $z = re^{i\theta}$ where

$$(6.4) \quad -if^*(-iz) = \sum_{k=0}^p (-1)^k A_k \frac{z^k + z^{k+1}}{(1-z)^{2k+1}}.$$

Long but straightforward and elementary calculations give

$$(6.5) \quad R_k(r, \theta) = \Re \left\{ \frac{z^k + z^{k+1}}{(1-z)^{2k+1}} \right\} \\ = \frac{(2k+1)! r^k}{(1-2r \cos \theta + r^2)^{2k+1}} \left[\frac{(-1)^k r^k (1-r^2)}{k!(k+1)!} + \sum_{m=1}^{k+1} D_m^{(k)} \cos m\theta \right],$$

where

$$(6.6) \quad (k+1-m)!(k+1+m)! D_m^{(k)} \\ = (-1)^{k-m} [(k+1-m)(r^{k-m} - r^{k+2+m}) + (k+1+m)(r^{k+m} - r^{k+2-m})].$$

Thus

$$(6.7) \quad \lim_{r \rightarrow 1} \frac{R_k(r, \theta)}{1-r^2} = \frac{(2k+1)!}{2^{2k+1}(1-\cos \theta)^{2k+1}} \left[\frac{(-1)^k}{k!(k+1)!} \right. \\ \left. + \sum_{m=1}^{k+1} \frac{2(-1)^{k-m-1}(m^2 - k - 1) \cos m\theta}{(k+1-m)!(k+1+m)!} \right]$$

$$\begin{aligned}
 &= \frac{(2k+1)!}{2^{2k+1}(1-\cos\theta)^{2k+1}} \left[\frac{(-2)^k(1-\cos\theta)^k(k\cos\theta+k+1)}{(2k+1)!} \right] \\
 &= \frac{(-1)^k(k\cos\theta+k+1)}{2^{k+1}(1-\cos\theta)^{k+1}}.
 \end{aligned}$$

For $z = re^{i\theta}$ we now have

$$\begin{aligned}
 (6.8) \quad &(1 - 2r\cos\theta + r^2)^{2p+1} \Re\{-if^*(iz)\} \\
 &= \sum_{k=0}^p (-1)^k A_k R_k(r, \theta) (1 - 2r\cos\theta + r^2)^{2p+1} = P(r, \theta).
 \end{aligned}$$

Here $P(r, \theta)$ is a polynomial in $\cos\theta$ of degree $2p$ and the number of changes of sign of $\Re f^*(z)$ is precisely the number of changes of sign of $P(r, \theta)$ on $|z| = r < 1$.

Since we shall see that

$$(6.9) \quad |a_n| = \sum_{k=0}^p \Delta(p, k, n) |a_k| \sim cn^{2p},$$

$\Re f^*(z)$ cannot change sign on $|z| = r$, for an interval $\rho < r < 1$, fewer than $2p$ times because of (1.9). Thus $P(r, \theta)$ changes sign at least $2p$ times on $|z| = r$, $\rho < r < 1$. We shall show next that $P(r, \theta)$ cannot vanish more than $2p$ times on $|z| = r$ for $\rho < r < 1$, $0 \leq \theta < 2\pi$. Let

$$\begin{aligned}
 (6.10) \quad P(\theta) &= \lim_{r \rightarrow 1} \frac{P(r, \theta)}{1 - r^2} \\
 &= \sum_{k=0}^p \frac{A_k}{2^{k+1}} \frac{(k\cos\theta + k + 1)}{(1 - \cos\theta)^{k+1}} \cdot 2^{2p+1} (1 - \cos\theta)^{2p+1} \\
 &= 2^{2p} (1 - \cos\theta)^p \sum_{k=0}^p \frac{A_k}{2^k} (k\cos\theta + k + 1) (1 - \cos\theta)^{p-k} \\
 &= 2^{2p} (1 - \cos\theta)^p Q(\theta),
 \end{aligned}$$

where

$$(6.11) \quad Q(\theta) = \sum_{k=0}^p \frac{A_k}{2^k} (k\cos\theta + k + 1) (1 - \cos\theta)^{p-k}$$

$$(6.12) \quad Q(\theta) > \frac{A_p}{2^p} > 0$$

provided not all $|a_0|, |a_1|, \dots, |a_p|$ are zero.

$P(\theta)$ is a polynomial of degree $2p$ in the variable $u = \cos\theta$, and has exactly p real zeros in the variable u in the range $-1 \leq u \leq 1$. Since $Q(\theta) > 0$, and because the zeros of any polynomial are continuous functions of its coefficients, we conclude that $P(r, \theta)$, as a polynomial of degree $2p$ in $u = \cos\theta$, also cannot have more than p real zeros u in the range $-1 \leq u \leq 1$. For, given ϵ sufficiently small but positive, and for values of r near one, $P(r, \theta)$ has exactly p complex zeros u_k in the circle $|u - 1| < \epsilon$, where u is regarded as a complex number,

$k = 1, 2, \dots, p$. Let

$$P(r, \theta) = P_1(u)P_2(u), \quad P_2(u) = \prod_{k=1}^p (u_k - u).$$

Then

$$\lim_{r \rightarrow 1} P_2(u) = (1 - u)^p, \quad \lim_{r \rightarrow 1} P_1(u) = 2^{2p} Q(\theta),$$

and when u is real, $-1 < u < 1$, $2^{2p} Q(\theta) > 2^p A_p > 0$. Thus, there exists a range $\rho < r < 1$ for which $|P_1(u)| > 2^{p-1} A_p > 0$, for u real and $-1 < u < 1$. It follows that, for a range $\rho < r < 1$, $P(r, \theta)$ has not more than p real zeros u in $-1 < u < 1$, and therefore not more than $2p$ zeros θ in the range $0 < \theta < 2\pi$. $P(r, \theta)$ therefore changes sign exactly $2p$ times on each circle $|z| = r$ for some range $\rho < r < 1$. This shows that $f^*(z)$ satisfies the hypothesis of Theorem 1.

In addition it is necessary to show that for $f^*(z)$ (6.9) holds. We note that the coefficient of z^n in $(z^k + iz^{k+1})(1 - iz)^{-2k-1}$ is

$$\frac{2n(n+k-1)!z^{n-k}}{(n-k)!(2k)!}, \quad n > k.$$

Thus for

$$\begin{aligned} f^*(z) &= \sum_{n=0}^{\infty} a_n z^n, \\ (6.13) \quad a_n &= i^{n+1} (2n) \sum_{k=0}^p \frac{(-1)^k (n+k-1)!}{(n-k)!(2k)!} A_k \\ &= 2ni^{n+1} \sum_{k=0}^p \frac{(-1)^k (n+k-1)!}{(n-k)!} \sum_{q=0}^k \frac{|a_q|}{(k+q)!(k-q)!} \\ &= 2ni^{n+1} \sum_{q=0}^p \sum_{k=q}^p \frac{(-1)^k (n+k-1)!}{(n-k)!(k+q)!(k-q)!} |a_q| \\ &= 2ni^{n+1} \sum_{q=0}^p \frac{(-1)^p (n+p)! |a_q|}{(n^2 - q^2)(n-p-1)!(p+q)!(p-q)!}, \\ (6.14) \quad |a_n| &= \sum_{q=0}^p \Delta(p, q, n) |a_q|, \end{aligned}$$

where $\Delta(p, q, n)$ is defined as in (1.9). Now the proof that (1.9) is sharp has been completed.

To show that (1.15) is also sharp we form the function

$$\begin{aligned} (6.15) \quad F^*(z) &= \int_0^z (f^*(z) - a_0) \frac{dz}{z} = \sum_1^{\infty} d_n z^n = \sum_1^{\infty} \frac{a_n}{n} z^n \\ &= \sum_{k=1}^p (-1)^{k+1} B_k \frac{z^k}{(1-iz)^{2k}} \end{aligned}$$

where

$$(6.16) \quad B_k = A_k - \frac{(2k)!}{k!k!} |a_0| = (2k)! \sum_{q=1}^k \frac{q|d_q|}{(k+q)!(k-q)!}.$$

It is immediately obvious from (6.14) and (6.15) that, for $F^*(z)$,

$$(6.17) \quad |d_n| = \sum_{q=1}^p \frac{q}{n} \Delta(p, q, n) |d_q|, \quad n > p,$$

which is to say that equality signs hold in (1.15). Further, since $F^*(z)$ is a polynomial of degree p in the variable $\zeta = iz/(1 - iz)^2$, and since ζ is a univalent function of z for $|z| < 1$, it follows at once that $F^*(z)$ is multivalent of order not exceeding p in $|z| < 1$. From (6.15) it is seen that for an interval $\rho < r < 1$ the derivative

$$(6.18) \quad \frac{\partial}{\partial \theta} \{ \Re F^*(re^{i\theta}) \}$$

changes sign exactly $2p$ times on $|z| = r$. From (6.17) it may be noted that for $F^*(z)$,

$$|d_n| \sim cn^{2p-1}.$$

Thus $F^*(z)$ cannot be multivalent of order $\nu < p$, for in that case we should conclude from a well-known result for the coefficients of a multivalent function of order ν that $d_n = O(n^{2\nu-1})$. Thus $F^*(z)$ is multivalent of order p in $|z| < 1$. $W = F^*(z)$ has the property that it maps $|z| = r$, $\rho < r < 1$, onto a contour such that every straight line parallel to the imaginary axis cuts it in at most $2p$ points.

REFERENCES

1. J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d'une variable complexe*, Ann. École Normale (3), vol. 48 (1931), 247-358.
2. A. W. Goodman, *On some determinants related to p -valent functions*, Trans. Amer. Math. Soc., vol. 63 (1948), 175-192.
3. A. W. Goodman and M. S. Robertson, *A class of multivalent functions*, Trans. Amer. Math. Soc., vol. 70 (1951), 127-136.
4. S. Kakeya, *On the function whose imaginary part on the unit circle changes its sign only twice*, Proc. Imp. Acad. Tokyo, vol. 18 (1942), 435-439.
5. M. S. Robertson, *Analytic functions star-like in one direction*, Amer. J. Math., vol. 58 (1936), 465-472.
6. ———, *A representation of all analytic functions in terms of functions with positive real parts*, Ann. Math., vol. 38 (1937), 770-783.
7. ———, *The variation of the sign of V for an analytic function $U + iV$* , Duke Math. J., vol. 5 (1939), 512-519.
8. W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z., vol. 35 (1932), 93-121.
9. O. Szász, *Über Funktionen, die den Einheitskreis schlicht abbilden*, Jber. dtsh. MatVer. vol. 42 (1932), 73-75.

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ANALYTIC FUNCTIONS WITH AN IRREGULAR LINEARLY MEASURABLE SET OF SINGULAR POINTS

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Introduction. V. V. Golubev, in his study [6], has constructed, by using definite integrals, various examples of analytic functions having a perfect nowhere-dense set of singular points. These functions were shown to be single-valued with a bounded imaginary part. In attempting to extend his work to the problem of constructing analytic functions having perfect, nowhere-dense singular sets under quite general conditions, he posed the following question: Given an arbitrary, perfect, nowhere-dense point-set E of positive measure in the complex plane, is it possible to construct, by passing a Jordan curve through E and by using definite integrals, an example of a single-valued analytic function, which has E as its singular set, with its imaginary part bounded.

In the present investigation, we shall require the set E , which is bounded and closed, to belong to the class of irregular sets of finite linear measure.¹ Hence, we wish to determine the possibility of obtaining, by using definite integrals, a function $\phi(z)$ having the following properties:

1. $\phi(z)$ analytic in the extended z -plane (except the points of an irregular, bounded and closed point-set E of essential singularities of positive linear measure);
2. $\phi(z)$ single-valued;
3. $|\Im \phi(z)|$ bounded.

The current problem is one that has evolved as a result of researches made by various authors. D. Pompeiu [10, pp. 914-915] was the first to exhibit an interest in constructing, with the help of definite integrals, an analytic function having a perfect, nowhere-dense, bounded set E of essential singular points. He proved that if E is of two-dimensional positive Lebesgue measure, there exist functions continuous and analytic in the extended z -plane with singularities in E .

Employing definite integrals, A. Denjoy [5, pp. 258-260] constructed an example of an analytic function $f(z)$ having a perfect, nowhere-dense set E of essential singularities of positive Lebesgue measure in the linear interval $0 < x < 1$. $f(z)$ was single-valued with a bounded imaginary part.

Golubev [6, p. 122] extended Denjoy's result to the case in which E was a perfect, nowhere-dense set of positive measure on any rectifiable curve and formed

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¹Point-sets of finite linear measure are divided into two classes: the first consisting of regular sets, and the second of irregular sets. Regular sets are completely analogous to rectifiable curves; irregular sets are dissimilar to rectifiable curves in fundamental geometrical properties. Cf. [1, pp. 424-426; 3, pp. 142-143].

the bounded function² $F(z) = e^{-if(z)}$. Moreover, he assumed the existence of any perfect, nowhere-dense set E of positive measure in the z -plane. He constructed, by passing a Jordan curve $g = g(t)$, $h = h(t)$ through E and by using definite integrals, an example of a single-valued analytic function, which has E as its singular set, with its imaginary part bounded. (The set E then corresponded to some perfect, nowhere-dense set E_t of values t .) Golubev, however, obtained questionable results.³

In §1 of this paper, we establish, for measurable functions defined and bounded on E , the general integral representation using Carathéodory linear measure. By means of the integral representation, we define, in §2, an analytic function as a function of its singular set. In §3, we generalize Golubev's technique of constructing a curvilinear integral of a function defined and continuous for a regular set E on any rectifiable curve to the case where E is an irregular set on any Jordan curve [6, p. 122]. We give, in §4, a characterization of the curvilinear integral and $\mathfrak{I}\phi(z)$.

1. Integral representation. We consider, in the complex plane, a point-set E satisfying the foregoing requirements. Let P denote any point of E , and $f(P)$ a single-valued function of a point defined and bounded on E . We enclose f in a finite or denumerable number of convex point-sets,

$$u_0, u_1, u_2, \dots, u_n, \dots$$

satisfying the following conditions:

- (a) $f(P)$, for each P , is interior to at least one of the sets u_0, u_1, u_2, \dots .
- (b) The diameter du_i of each u_i is smaller than a positive number ρ chosen in advance.⁴

Now $f(P)$ is measurable with regard to the covering u_i , that is,⁵ $E(|f| > \mu) = E(f > \mu)$, for $\mu > 0$, forms a measurable set⁶ E_μ of points P . We insert between

²The analytic function $F(z)$ is single-valued and bounded in a domain whose boundary in a regular set E of positive measure. According to Fatou's Theorem, as generalized by Golubev, $F(z)$ has on nearly all points of E a definite value, a fact from which we conclude E to be removable. [6, p. 44; 8, pp. 154-157; 11, p. 40].

³Golubev [6, pp. 127-129] noted that his construction of this function was weakened because E_t had no direct connection with E , and could be selected arbitrarily.

⁴The convex point-set u_i is orthogonally projected on a plane, the result of which is another convex point-set whose area depends upon the position of the particular plane in space. We call the upper bound of the areas for all possible planes the two-dimensional Carathéodory diameter du_i of u_i [4, p. 426; 7, p. 162].

⁵ $E(|f| > \mu)$ is the sum of two measurable sets $E(f > \mu)$ and $E(f < -\mu)$. However, the latter set does not exist for the measure under consideration. ($E(|f| > \mu)$ denotes the subset of E for which $|f(P)| > \mu$.)

⁶A plane set E is called measurable with respect to Carathéodory linear measure if the relation

$$L_1(A + B) = L_1(A) + L_1(B)$$

for every pair of sets A and B contained in E and its complement, $C(E)$ [4, pp. 404-426].

the upper bound M and the lower bound m of f the following numbers;

$$\mu_1 < \mu_2 < \mu_3 < \dots < \mu_{n-1},$$

and establish, for the integral of $f(P)$ over E , the representation,

$$\int_E f(P) du = \liminf_{\rho \rightarrow 0} \sum_{i=0}^n \mu_i du_i \quad (\mu_0 = m, \mu_n = M).$$

E is a bounded point-set; hence, as ρ approaches zero, there exists a finite limit. We denote the linear measure of f by $L_2(f)$, where⁷

$$L_2(f) = \int_E du_i = \liminf_{\rho \rightarrow 0} \sum_{i=0}^n du_i.$$

2. Analytic function as function of its singular set. The integral of $f(P)$, by the nature of its construction, is a number that depends upon the point-set E . This function of a set,

$$F(E) = \int_E f(P) du$$

is defined as the definite integral of $f(P)$ over E , and

$$|F(E)| = \left| \int_E f(P) du \right| < \int_E |f(P)| du < ML_2(f),$$

where M denotes the maximum value of $f(P)$ over E .

We consider now the function

$$\phi(z) = \int_E f(P, z) du,$$

a function defined by a definite integral which contains in the integrand a parameter. Let $f(P, z)$ be a single-valued, bounded function defined when P lies in E and z in the complementary set, $C(E)$. We first prove an extension of a well-known integral theorem in the Theory of Functions to integrals of the class under consideration. The theorem will serve as a nucleus for current developments.

THEOREM 1. *If $f(P, z)$ is a continuous function of P and z together, the function*

$$\phi(z) = \int_E f(P, z) du$$

is continuous in $C(E)$.

Moreover, if $f(P, z)$ has for each z a partial derivative $f_z(P, z)$ continuous in P and z together, the function $\phi(z)$ is analytic in $C(E)$, that is,

$$\phi'(z) = \int_E f_z(P, z) du.$$

⁷The measure $L_2(f)$ is non-negative and single-valued. For the conditions that $L_2(f)$ must satisfy, see [7, pp. 158-159].

The first part of the theorem is valid because of the continuity of $f(P, z)$ in the two variables (P, z) .

From the existence and continuity of the partial derivative $f_z(P, z)$, we derive the continuity of the partial derivatives $f_x(P, z)$ and $f_y(P, z)$. Further, these derivatives satisfy the Cauchy-Riemann differential equations. We have

$$f_x(P, z) = -if_y(P, z) = f_z(P, z)$$

and therefore $\phi(z)$ possesses a partial derivative with respect to x and y continuous in $C(E)$. Hence,

$$\phi_z(z) = \int_E f_z(P, z) du = -i \int_E f_y(P, z) du = -i \phi_y(z).$$

These derivatives likewise fulfil the requirements of the Cauchy-Riemann differential equations. Consequently, $\phi(z)$ is analytic in $C(E)$, and

$$\phi'(z) = \int_E f_z(P, z) du.$$

This proves the second part.

In compliance with the hypothesis of Theorem 1 we select

$$f(P, z) = \frac{1}{P - z}$$

and construct the function

$$\phi(z) = \int_E \frac{du}{P - z}.$$

The integral in the right member is a function of an irregular set of singular points in a sense analogous to that in which a function of a set of singular lines has been constructed with the help of definite integrals.⁸ Thus, $\phi(z)$ represents at most a denumerable set of functions analytic except for certain singular points.

In $C(E)$, $\phi(z)$ is an analytic function. Its first derivative is given by the formula

$$\phi'(z) = \int_E \frac{du}{(P - z)^2}.$$

Moreover, $\phi(z)$ possesses derivatives of every order analytic in $C(E)$ and they are given by the formulae

$$\phi''(z) = 2! \int_E \frac{du}{(P - z)^3}$$

and, in general,

$$\phi^{(n)}(z) = n! \int_E \frac{du}{(P - z)^{n+1}} \quad (n = 1, 2, 3, \dots).$$

From the well-known fact that a function can be represented by a power

⁸For examples of functions of a set of singular lines, cf. [6, pp. 92-97].

series in the neighbourhood of any point of a domain in which it is analytic, we have the following result:

COROLLARY 1. *If $z = z_0$ be any fixed point in $C(E)$,*

$$\phi(z) = \int_E \frac{du}{P - z}$$

can be represented, in a certain neighbourhood of this point, by a Taylor series. This series will converge and represent the function throughout the largest circle, about $z = z_0$ as centre, which contains in its interior no point of E .

We determine the nature of the function

$$\phi(z) = \int_E \frac{du}{P - z}$$

in the neighbourhood of the point $z = \infty$. Let us begin by making the transformation $z' = 1/z$, writing $\Psi(z') = \phi(1/z')$ and examining the transformed function $\Psi(z')$ for values in the neighbourhood of $z' = 0$. First, we have

$$\Psi(z') = \int_E \frac{z' du}{Pz' - 1},$$

and $\Psi(0) = 0$. We next take successive derivatives of $\Psi(z')$, then place $z' = 0$, obtaining

$$\Psi'(z') = - \int_E \frac{du}{(Pz' - 1)^2}, \quad \Psi'(0) = - \int_E du,$$

$$\Psi''(z') = 2! \int_E \frac{P du}{(Pz' - 1)^3}, \quad \Psi''(0) = - 2! \int_E P du,$$

and, in general,

$$\Psi^{(n)}(z') = (-1)^n n! \int_E \frac{P^{n-1} du}{(Pz' - 1)^{n+1}}, \quad \Psi^{(n)}(0) = - n! \int_E P^{n-1} du$$

($n = 1, 2, 3, \dots$).

The series

$$\Psi(z') = \Psi(0) + \Psi'(0)z' + \frac{\Psi''(0)z'^2}{2!} + \dots + \frac{\Psi^{(n)}(0)z'^n}{n!} + \dots$$

becomes, by expressing Ψ and its derivatives in terms of integrals,

$$\Psi(z') = - \left(\int_E z' du + \int_E Pz'^2 du + \dots + \int_E P^{n-1} z'^n du + \dots \right).$$

Then

$$\phi(z) = - \left(\int_E z^{-1} du + \int_E Pz^{-2} du + \dots + \int_E P^{n-1} z^{-n} du + \dots \right).$$

The appearance of negative powers of z in the right member indicates that $\phi(z)$ is analytic in the neighbourhood of the point $z = \infty$, and the absence of

the constant term shows that the function has a root at infinity. We have proved

THEOREM 2.

$$\phi(z) = \int_E \frac{du}{P - z}$$

is analytic at $z = \infty$, and $\phi(\infty) = 0$.

An examination of the single-valued character of $\phi(z)$ discloses a question: Does $\phi(z)$ return to its original value when z describes a continuous closed path around E ? Let us enclose E in the smallest rectangle R which contains the point set in its interior. The domain exterior to R we designate by S ; $\phi(z)$ is analytic in S . Moreover, S is simply connected. Therefore, we have an analytic function in a simply connected domain; a fact which proves, according to the monodromy theorem, that $\phi(z)$ is single-valued for any closed path described by z around E .

A second question confronts us: Can $\phi(z)$ be analytically continued through E ? Let us consider a path K in $C(E)$ which begins at a point z_0 and separates E into two distinct proper subsets E_1 and E_2 , each of which has positive linear measure. The path K , which satisfies this condition, is known to exist because of the density classification by which irregular sets are defined.¹ By Corollary 1, $\phi(z)$ can be represented in a certain neighbourhood of z_0 by a power series which will converge and represent a functional element within the largest circle, centre z_0 , which contains in its interior no point of E . We continue this element of $\phi(z)$ along K through power series expansions by choosing points, as centres of circles of convergence, along the path in such a way that the circles form a chain. Continuation by this means is possible through repeated application of the corollary and the use of the identity theorem for analytic functions. We observe from the foregoing remarks that the analytic continuation of $\phi(z)$ along K is everywhere feasible.

We form the function $\phi(z) = \phi_1(z) + \phi_2(z)$, where

$$\phi_1(z) = \int_E \frac{du_1}{P - z}, \quad \phi_2(z) = \int_E \frac{du_2}{P - z},$$

and choose for z a path which includes k in the following manner: from z_0 , z passes along k , encircles E_2 by a counterclockwise movement, continues a circuit around E_1 to k , and finally reverses its direction along this path to z_0 . The path thus described by z is equivalent to a single continuous closed circuit around E , a case for which we have shown $\phi(z)$ to be single-valued. Hence

$$\phi(z) = \int_E \frac{du_1}{P - z} + \int_E \frac{du_2}{P - z} = \int_E \frac{du}{P - z},$$

and we have proved

THEOREM 3. *The analytic function*

$$\phi(z) = \int_E \frac{du}{P - z}$$

is single-valued in $C(E)$ if E is an irregular point-set.

We shall prove [6, pp. 128-129]

THEOREM 4. *The point-set E is a singular set for $\phi(z)$.*

Let us assume that $\phi(z)$ is constant. In $C(E)$, $\phi(z)$ is a single-valued analytic function, which is attested by Theorem 3. According to Theorem 2, $\phi(z) = 0$ for $z = \infty$. Therefore, if $\phi(z)$ is constant, $\phi(z) = 0$ and $z\phi(z) = 0$. However,

$$\lim_{z \rightarrow \infty} z\phi(z) = \lim_{z \rightarrow \infty} \int_E \frac{du}{(P/z) - 1} = - \int_E du = -L_2(f).$$

We thus have a contradiction since the measure $L_2(f)$ is known to be non-negative. We conclude that at least some points of E are singular for $\phi(z)$. Let us denote these singular points by E_1 , and represent E as the sum

$$E = E_1 + E_{11}.$$

Then

$$\phi(z) = \phi_1(z) + \phi_{11}(z),$$

where $\phi_1(z)$ is the function which has E_1 as its singular set. Applying the foregoing procedure to $\phi_{11}(z)$, we show that at least some points of E_{11} are singular for $\phi_{11}(z)$, for example, E_2 . We now represent E_{11} as the sum

$$E_{11} = E_2 + E_{12},$$

and

$$\phi_{11}(z) = \phi_2(z) + \phi_{12}(z),$$

where $\phi_2(z)$ is the function which has E_2 as its singular set. We continue this process and obtain

$$E = E_1 + E_2 + E_3 + \dots$$

singular sets for

$$\phi(z) = \phi_1(z) + \phi_2(z) + \phi_3(z) + \dots$$

respectively, remembering that sets of measure zero correspond to isolated singular points. We mean, by this, that the characteristics of functions having a set of singular points of measure zero compare very closely to those having isolated essential singular points.⁹ This completes the proof.

3. The curvilinear integral. We come to the third property, namely, to determine whether or not the function $\phi(z)$ has a bounded imaginary part. We consider the construction of an expression analogous to a curvilinear integral of $f(P)$ along E .

According to the researches of Besicovitch [1, p. 455], there exists a finite or denumerable set G of Jordan curves which contain almost all points of E . We denote by J_1 the curve of G which contains a subset $'E$ of E such that the linear measure of the plane set $'E$ satisfies the inequality $L('E) > L(E) - \epsilon$, ϵ being a positive number.

The set $'E$ is a closed subset of E . Therefore, the $C('E)$ complementary to

⁹For details concerning this matter, cf. [6, pp. 126-127].

$$\int_{E_1} |f(P)| |e^{i\theta} du_2| < M \liminf_{\rho \rightarrow 0} \sum_m \sum_n |e^{i\theta_m} du_{2m}^n|.$$

In accordance with the meaning of the double sums of the right members, and by making use of the triangle theorem, we have

$$M \liminf_{\rho \rightarrow 0} \sum_m \sum_n |e^{i\theta_n} du_m^n| < M \liminf_{\rho \rightarrow 0} \sum_m \sum_n (|e^{i\theta_{1n}} du_{1m}^n| + |e^{i\theta_{2n}} du_{2m}^n|),$$

if $\theta \neq \theta_1 \neq \theta_2 \neq 0$. If $\theta = \theta_1 = 0$, the equality sign holds for the same reasons.

THEOREM 6. *If ${}_+'E$ and ${}_-'E$ denote two opposite directions in which the integral is taken along $'E$,*

$$\int_{+'E} f(P) |e^{i\theta} du| = \int_{-'E} f(P) |e^{i\theta} du|,$$

i.e., the value of the integral is independent of the direction of integration.

The proof of this Theorem follows immediately from the definition of the integral.

THEOREM 7. *The curvilinear integral is dependent upon the particular subset $^{(n)}E$ of E through which a Jordan curve passes.*

According to a theorem of Besicovitch [1, p. 455], there exists a finite or denumerable set of Jordan curves that can be passed through the points of E . The intersection of each Jordan curve with E is of positive measure. The curve J_1 , as we have indicated in §3, contains a subset $'E$ of E . We denote by $J_2, J_3, \dots, J_n, \dots$ those Jordan curves that contain subsets $''E, ''', E, \dots, ^{(n)}E, \dots$ of E respectively. Through the application of procedures and operations used in §3, the curvilinear integral of $f(P)$ along each $^{(n)}E$ ($n = 2, 3, \dots$) can be easily shown to exist. The Jordan curves J_n are assumed to be distinct. This proves the theorem.

We resolve

$$\int_{+'E} f(P) |e^{i\theta} du|$$

into real and imaginary parts. We have shown that

$$\int_{+'E} f(P) |e^{i\theta} du| = \liminf_{\rho \rightarrow 0} \sum_m \sum_n f(P_m^n) |e^{i\theta_n} du_m^n|.$$

Now let¹²

$$f(P) = f_1(P_1) + if_2(P_1).$$

Then

$$\begin{aligned} \liminf_{\rho \rightarrow 0} \sum_m \sum_n f(P_m^n) |e^{i\theta_n} du_m^n| &= \liminf_{\rho \rightarrow 0} \sum_m \sum_n f_1(P_{1m}^n) |e^{i\theta_n} du_m^n| \\ &\quad + \liminf_{\rho \rightarrow 0} \sum_m \sum_n f_2(P_{1m}^n) |e^{i\theta_n} du_m^n|. \end{aligned}$$

¹² $f_1(P_1)$ and $f_2(P_1)$ are real functions of the point P_1 .

$f(P)$ is continuous along $'E$. Therefore,

$$\int_{\cdot E} f(P) |e^{\theta} du| = \int_{\cdot E} f_1(P_1) |e^{\theta} du| + i \int_{\cdot E} f_2(P_1) |e^{\theta} du|.$$

The two integrals on the right, consistent with their meaning, represent a generalization of curvilinear integrals of functions of real variables.

We form the function

$$\phi(z) = \int_{\cdot E} \frac{|e^{\theta} du|}{P - z},$$

where z is in $C('E)$, and investigate the character of $\Im \phi(z)$. For this purpose, we let $P = a + ib$. Then

$$\begin{aligned} \phi(z) &= \int_{\cdot E} \frac{|e^{\theta} du|}{P - z} = \int_{\cdot E} \frac{(a - x) |e^{\theta} du| - i(b - y) |e^{\theta} du|}{(a - x)^2 + (b - y)^2} \\ \Im \phi(z) &= \int_{\cdot E} \frac{(y - b) |e^{\theta} du|}{(a - x)^2 + (b - y)^2}. \end{aligned}$$

There exists no positive number M which is not exceeded¹³ by $|\Im \phi(z)|$ for any z in $C('E)$. We conclude that E is not a removable point-set.

With reference to the classical Riemann theorem on removable singularities of an analytic function, Besicovitch [2, p. 2] has shown that a removable set of singularities cannot be an arbitrary set of positive linear measure. The results which we have obtained indicate that removable sets cannot be irregular sets. This means that removable sets are restricted to the class of regular sets of positive linear measure.

We have demonstrated the possibility of inspecting an analytic function as a function of its set of singular points. If we take this point of view, the core of the study of analytic functions may well shift from the domain in which the function is analytic to the set of its singular points. The question as to whether or not it was possible to examine an analytic function in this manner was raised by Golubev [6, pp. 156-157] at the close of his study.

In our extension of the study of Golubev to the case of irregular sets of positive measure, we have constructed an analytic function satisfying two of the three properties which we proposed for investigation. We summarize our work with the following theorem:

THEOREM 8. *Let E be a bounded and closed point-set in the complex plane. If E is an irregular, linearly measurable point-set, there exists in the neighbourhood of E a function $\phi(z)$, single-valued and analytic in the extended plane, with E as its singular set.*

¹³The assertion in the text can be easily verified by replacing $|e^{\theta} du|$ by its analogue ds .

REFERENCES

1. A. S. Besicovitch, *On the geometrical properties of linearly measurable sets*, Math. Ann., vol. 98 (1927), 422-464.
2. ———, *On conditions for a function to be analytic*, Proc. London Math. Soc. (2), vol. 32 (1931), 1-9.
3. A. S. Besicovitch and G. Walker, *On the density of irregular linearly measurable sets of points*, Proc. London Math. Soc. (2), vol. 32 (1931), 142-153.
4. C. Carathéodory, *Über das lineare Mass von Punktmengen*, Nachr. Ges. Wiss. Göttingen (1914), 404-426.
5. A. Denjoy, *Sur les fonctions analytiques uniformes à singularités discontinues*, C. R. Acad. Sci., Paris, vol. 149 (1909), 258-260.
6. V. V. Golubev, *Single-valued analytic functions with perfect singular sets* (In Russian), Bulletin of the University of Moscow (1916), 1-157.
7. F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann., vol. 79 (1919), 157-167.
8. N. Lusin and J. Priwaloff, *Sur l'unicité et la multiplicité des fonctions analytiques*, Ann. sc. Éc. norm. sup. Paris (3), vol. 42 (1925), 154-157.
9. R. Nevanlinna, *Eindeutige analytische Functionen* (Berlin, 1936), 106-142.
10. D. Pompeiu, *Sur les singularités des fonctions analytiques uniformes*, C. R. Acad. Sci., Paris, vol. 139 (1904), 914-915.
11. F. and M. Riesz, *Über die Randwerte von analytischen Functionen*, Compte Rendu Quatrième Congrès des Mathématiciens Scandinaves (1916), 40.

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A NEW REPRESENTATION AND INVERSION THEORY FOR THE LAPLACE TRANSFORMATION

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1. Introduction. In the literature, considerable attention has been devoted to the study of inversion operators for the Laplace transformation. In particular, much interest attaches to "real" inversion operators, i.e., operators which make use of the values of the generating function arising only from real values of the independent variable. Several of these operators are known (see for example Widder [3, chap. 7, §6; chap. 8, §25], Hirschman [2]).

In this paper we shall develop the inversion and representation theory for a new "real" inversion operator. If¹

$$\text{I} \quad f(s) = \int_0^{\infty} e^{-st} \phi(t) dt,$$

and

$$\text{II} \quad L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1}) \int_0^{\infty} x^{-1} \cos(2kx^1) f(k(x+1)/t) dx,$$

then we shall show that under certain conditions

$$\lim_{k \rightarrow \infty} L_{k,t}[f(s)] = \phi(t).$$

This operator was given by A. Erdélyi [1]. However, the resulting inversion and representation theory were not developed there.

There is another operator related to II which is given by

$$L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1}) \int_0^{\infty} \sin(2kx^1) f(k(x+1)/t) dx.$$

This operator and the operator II are special cases of another operator,

$$L_{k,t}[f(s)] = [2tK_s(2k)]^{-1} k \int_0^{\infty} x^{1/2} J_s(2kx^1) f(k(x+1)/t) dx,$$

which is also given in Professor Erdélyi's paper. The inversion and representation theory for this last operator has also been investigated by the author, and it was found that the resulting theory is similar in every respect to that for the operator II.

The operator II has some points of resemblance to Phragmén's operator [3, chap. 7, §2] in that both are "real," involve only the values of $f(s)$ for large values of s , and involve only elementary functions. Unlike Phragmén's operator though, II is an integral operator.

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¹The notations introduced at this point will be used consistently throughout the rest of this paper.

To avoid unessential difficulties we shall restrict our discussion to the operator II. We shall show that the operator II inverts the transformation provided only that $e^{-st}t^{-1}\phi(t)$ is absolutely integrable from zero to infinity for some value of s . Preliminary to the representation theory we shall show that

$$\lim_{k \rightarrow \infty} \int_0^{\infty} e^{-\sigma t} L_{k,t}[f(s)] dt = f(\sigma)$$

under the conditions that $s^{-1}f(s) \in L(\delta, \infty)$ for all positive δ , that

$$\int_x^{\infty} y^{-1}f(y^{-1})dy$$

be suitably restricted in its behaviour at zero and infinity, and that

$$e^{-\sigma t} L_{k,t}[f(s)] \in L(0, \infty)$$

for some σ . Representation theorems are then established for $L_{k,t}[f(s)]$ in various classes of functions. Lastly the Laplace-Stieltjes transformation is treated in a similar manner.

This paper embodies the results of a portion of a study being carried out on the inversion of the Laplace transformation. Other portions of the study deal with the Laplace transformation of Banach-valued functions of a real variable.

2. Inversion theorem.

THEOREM 1. *If $e^{-st}t^{-1}\phi(t) \in L(0, \infty)$ for all $s > \gamma$, then $f(s)$ exists for $s > \gamma$, and for $t > 0$,*

$$(i) \quad \lim_{k \rightarrow \infty} L_{k,t}[f(s)] = \frac{1}{2} \{ \phi(t+) + \phi(t-) \}$$

at every point at which $\phi(t+)$ and $\phi(t-)$ both exist.

$$(ii) \quad \lim_{k \rightarrow \infty} L_{k,t}[f(s)] = \phi(t)$$

at every point of the Lebesgue set of $\phi(t)$. *

Proof. We shall use Widder [3, p. 25, theorem 15c; pp. 278-280, theorem 2b and corollaries]. Operating formally we have

$$\begin{aligned} L_{k,t}[f(s)] &= (ke^{2k}(\pi t)^{-1}) \int_0^{\infty} x^{-1} \cos(2kx^{1/2}) f(k(x+1)/t) dx \\ &= (ke^{2k}(\pi t)^{-1}) \int_0^{\infty} x^{-1} \cos(2kx^{1/2}) dx \int_0^{\infty} e^{-k(x+1)u/t} \phi(u) du \\ &= (ke^{2k}(\pi t)^{-1}) \int_0^{\infty} e^{-kx^{1/2}} \phi(u) du \int_0^{\infty} e^{-kxu/t} x^{-1} \cos(2kx^{1/2}) dx \\ &= (2k^{1/2}e^{2k}(\pi t^{1/2})^{-1}) \int_0^{\infty} e^{-kx^{1/2}} u^{-1} \phi(u) du \int_0^{\infty} e^{-v^2} \cos\{2(kt/u)^{1/2}v\} dv \\ &\quad \text{(where } v^2 = kxu/t) \end{aligned}$$

$$\begin{aligned}
&= (k(\pi t)^{-1})^{\frac{1}{2}} e^{2k} \int_0^{\infty} e^{-k(u t^{-1} + u^{-1})} u^{-\frac{1}{2}} \phi(u) du \\
&\sim (k(\pi t)^{-1})^{\frac{1}{2}} e^{2k} (t^{\frac{1}{2}}/k)^{\frac{1}{2}} \phi(t) t^{-\frac{1}{2}} e^{-2k} \Gamma(\frac{1}{2}) \\
&= \phi(t) \qquad \text{as } k \rightarrow \infty.
\end{aligned}$$

If the conditions of the above-mentioned theorems are fulfilled, i.e., the conditions for the interchange of integrations and for the asymptotic evaluation, the theorem will be proved.

All the conditions for the interchange of integrations are fulfilled except possibly

$$\int_0^{\infty} e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du \int_0^{\infty} |e^{-sv} \cos(2(kt/u)^{\frac{1}{2}} v)| dv < \infty.$$

However, this last condition is also fulfilled since

$$\int_0^{\infty} |e^{-sv} \cos(2(kt/u)^{\frac{1}{2}} v)| dv \leq \int_0^{\infty} e^{-sv} dv = \frac{1}{s} \sqrt{\pi},$$

and thus

$$\int_0^{\infty} e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du \int_0^{\infty} |e^{-sv} \cos(2(kt/u)^{\frac{1}{2}} v)| dv \leq \frac{1}{s} \sqrt{\pi} \int_0^{\infty} e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du < \infty$$

if $k/t > \gamma$. To justify the asymptotic evaluation, it must be shown that

$$I_1 = \left| k^{\frac{1}{2}} e^{2k} \int_0^{t-\frac{1}{2}} e^{-k(u t^{-1} + u^{-1})} u^{-\frac{1}{2}} \phi(u) du \right|,$$

and

$$I_2 = \left| k^{\frac{1}{2}} e^{2k} \int_{t+\frac{1}{2}}^{\infty} e^{-k(u t^{-1} + u^{-1})} u^{-\frac{1}{2}} \phi(u) du \right|$$

tend to zero as $k \rightarrow \infty$.

For the latter, choose $k_0 > \gamma t$. Then for $k > k_0$,

$$\begin{aligned}
I_2 &\leq k^{\frac{1}{2}} e^{2k} \int_{t+\frac{1}{2}}^{\infty} e^{-k(u t^{-1} + u^{-1})} |u^{-\frac{1}{2}} \phi(u)| du \\
&= k^{\frac{1}{2}} e^{2k} \int_{t+\frac{1}{2}}^{\infty} e^{-k_0(u t^{-1} + u^{-1})} e^{-(k-k_0)(u t^{-1} + u^{-1})} |u^{-\frac{1}{2}} \phi(u)| du \\
&\leq k^{\frac{1}{2}} e^{2k} e^{-(k-k_0)(t(t+\frac{1}{2})^{-1} + (t+\frac{1}{2}) t^{-1})} \int_{t+\frac{1}{2}}^{\infty} e^{-k_0(u t^{-1} + u^{-1})} |u^{-\frac{1}{2}} \phi(u)| du \\
&= A(t) k^{\frac{1}{2}} e^{2k_0} e^{-(k-k_0)[t^2 + (t+\frac{1}{2})^2 - 2t(t+\frac{1}{2})]} / [t(t+\frac{1}{2})] \\
&= B(t) k^{\frac{1}{2}} e^{-k^2 \delta / (t(t+\frac{1}{2}))} \rightarrow 0 \qquad \text{as } k \rightarrow \infty \text{ for } t > 0.
\end{aligned}$$

Similarly $I_1 \rightarrow 0$ as $k \rightarrow \infty$.

3. Some lemmas. In this section we shall prove some lemmas that will be needed in the representation theory.

LEMMA 1. If

$$(1) \quad s^{-1}f(s) \in L(\delta, \infty) \quad \text{for all } \delta > 0,$$

$$(2) \quad \begin{aligned} F(x) &= \int_x^\infty y^{-1}|f(y)|dy = O(x^{-m}) \quad \text{with } m > 0, \text{ as } x \rightarrow \infty, \\ F(x) &= O(e^{\gamma/x}) \quad \text{with } \gamma > 0, \text{ as } x \rightarrow 0+, \end{aligned}$$

$$(3) \quad m + n > 0,$$

then

$$(i) \quad s^{n-1}f(s^{-1}) \in L(0, R) \quad \text{for all } R > 0,$$

$$(ii) \quad G(t) = \int_0^t u^{n-1}|f(u^{-1})|du = O(t^{m+n}) \quad \text{as } t \rightarrow 0+,$$

$$(iii) \quad \begin{aligned} G(t) &= O(t^n e^{\gamma t}) \quad \text{as } t \rightarrow \infty \text{ if either } \gamma > 0 \text{ or } n > 0 \\ &= O(1) \quad \text{as } t \rightarrow \infty \text{ if } \gamma = 0 \text{ and } n < 0, \end{aligned}$$

$$(iv) \quad \int_0^\infty e^{-st} t^{n-1} f(t^{-1}) dt$$

is absolutely convergent for $s > \gamma$, and is $O(s^{-m-n})$ as $s \rightarrow \infty$.

Proof. (i) Clearly $u^{n-1}|f(u^{-1})| \in L(\delta, R)$ for all $R > \delta > 0$. Thus

$$\begin{aligned} \int_\epsilon^t u^{n-1}|f(u^{-1})|du &= \int_\epsilon^t u^n dF(u^{-1}) \\ &= t^n F(t^{-1}) - \epsilon^n F(\epsilon^{-1}) - n \int_\epsilon^t u^{n-1} F(u^{-1}) du, \end{aligned}$$

and by (2), the right-hand side tends to a finite limit as $\epsilon \rightarrow 0$ since $m + n > 0$. Thus

$$G(t) = t^n F(t^{-1}) - n \int_0^t u^{n-1} F(u^{-1}) du.$$

$$(ii) \quad G(t) = t^n F(t^{-1}) - n \int_0^t u^{n-1} F(u^{-1}) du = O(t^{m+n})$$

by the last equation and (2).

(iii) Let $m + n > 0$ and either $\gamma > 0$ or $n > 0$ so that

$$\int_0^t u^{n-1} F(u^{-1}) du \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

$$G(t) = t^n O(e^{\gamma t}) - n \int_0^t u^{n-1} O(e^{\gamma u}) du = O(t^n e^{\gamma t}).$$

If $\gamma = 0$, $n < 0$, $G(t)$ is clearly bounded.

(iv) Clearly $e^{-st} t^{n-1}|f(t^{-1})| \in L(\delta, R)$ for all $R > \delta > 0$.

$$\int_\delta^R e^{-st} t^{n-1}|f(t^{-1})|dt = e^{-sR} G(R) - e^{-s\delta} G(\delta) + s \int_\delta^R e^{-st} G(t) dt.$$

Convergence as $\delta \rightarrow 0$ follows from (ii) since $m + n > 0$; convergence as $R \rightarrow \infty$ follows from (iii) if $s > \gamma$. Moreover, from Widder [1, p. 181, theorem 1], the integral is $O(s^{-m-n})$ as $s \rightarrow \infty$ (and is $O((s - \gamma)^{-n})$ as $s \rightarrow \gamma +$ if $n > 0$).

LEMMA 2. If $f(s)$ satisfies the requirements of Lemma 1 with $m > \frac{1}{2}$ then, for each $k > 0$, $L_{k,t}[f(s)]$ exists for almost all $t > 0$. In particular, $L_{k,t}[f(s)]$ exists when $k, t > 0$ and k/t is in the Lebesgue set of $f(s)$.

Proof. We have to show convergence of the integral II,

$$L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1}) \int_0^\infty x^{-1} \cos(2kx) f(k(x+1)/t) dx,$$

at the origin and at infinity.

If k/t is in the Lebesgue set of $f(s)$, we have

$$J(h) = \int_0^h |f(k(x+1)/t) - f(k/t)| dx = o(h), \quad h \rightarrow 0.$$

Thus

$$\begin{aligned} \int_0^h x^{-1} \cos(2kx) f(k(x+1)/t) dx \\ &< \int_0^h x^{-1} dx f(k/t) + \int_0^h x^{-1} |f(k(x+1)/t) - f(k/t)| dx \\ &= o(1) + \int_0^h x^{-1} dJ(x) = o(1) + \frac{1}{2} \int_0^h x^{-3/2} J(x) dx \\ &= o(1) + \frac{1}{2} \int_0^h x^{-3/2} o(x) dx = o(1) \end{aligned}$$

as $\epsilon, \delta \rightarrow 0+$, and II converges absolutely at the origin.

From Lemma 1 we have

$$\int_0^\epsilon u^{-3/2} |f(u^{-1})| du < \infty.$$

Here we put $u^{-1} = k(x+1)/t$ and choose $\epsilon < t/k$. We then have

$$(k/t)^{1/2} \int_{(t/(k\epsilon)-1)}^\infty (1+x)^{-1} |f(k(x+1)/t)| dx < \infty$$

and thus II converges absolutely at infinity.

4. Fundamental theorem.

THEOREM 2. If

$$(1) \quad s^{-1}f(s) \in L(\delta, \infty) \quad \text{for all } \delta > 0,$$

$$(2) \quad F(x) = \int_x^\infty y^{-1} |f(y)| dy = O(x^{-m}) \quad \text{with } m > \frac{1}{2}, \text{ as } x \rightarrow \infty,$$

$$F(x) = O(e^{\gamma/x}), \quad \text{with } \gamma > 0, \text{ as } x \rightarrow 0+,$$

$$(3) \quad e^{-\sigma t} L_{k,1}[f(s)] \in L(0, \infty), \quad \sigma > \gamma_1 \text{ for all } k > k_0,$$

then

$$\lim_{k \rightarrow \infty} \int_0^{\infty} e^{-\sigma t} L_{k,1}[f(s)] dt = f(\sigma)$$

at every point of the Lebesgue set of $f(\sigma)$, $\sigma > \gamma_1$.

Proof. $L_{k,1}[f(s)]$ exists and has a Laplace transform when $\sigma > \gamma_1$. To prove the assertion we shall use the same theorems of Widder [3], that were used in the proof of Theorem 1. Operating formally we have,

$$\begin{aligned} \int_0^{\infty} e^{-\sigma t} L_{k,1}[f(s)] dt &= (ke^{2k}/\pi) \int_0^{\infty} e^{-\sigma t} t^{-1} dt \int_0^{\infty} x^{-1} \cos(2kx^3) f(k(x+1)/t) dx \\ &= (2ke^{2k}/\pi) \int_0^{\infty} e^{-\sigma t} t^{-1} dt \int_0^{\infty} \cos(2ky) f(k(y^2+1)/t) dy \quad (\text{where } y^2 = x) \\ &= (2ke^{2k}/\pi) \int_0^{\infty} \cos(2ky) dy \int_0^{\infty} e^{-\sigma t} t^{-1} f(k(y^2+1)/t) dt \\ &= (2ke^{2k}/\pi) \int_0^{\infty} \cos(2ky) dy \int_0^{\infty} e^{-k\sigma u(y^2+1)} u^{-1} f(u^{-1}) du \quad (\text{where } u^{-1} = k(y^2+1)/t) \\ &= (2ke^{2k}/\pi) \int_0^{\infty} e^{-k\sigma u} u^{-1} f(u^{-1}) du \int_0^{\infty} e^{-k\sigma u y^2} \cos(2ky) dy \\ &= (2k^{\frac{1}{2}} e^{2k} (\pi \sigma^{\frac{1}{2}})^{-1}) \int_0^{\infty} e^{-k\sigma u} u^{-3/2} f(u^{-1}) du \int_0^{\infty} e^{-v^2} \cos\{2(k\sigma u)^{-1/2} v\} dv \\ &\quad (\text{where } v^2 = k\sigma u y^2) \\ &= e^{2k} (k(\sigma \pi)^{-1})^{\frac{1}{2}} \int_0^{\infty} e^{-k(\sigma u + (\sigma u)^{-1})} u^{-3/2} f(u^{-1}) du \rightarrow f(\sigma) \quad \text{as } k \rightarrow \infty. \end{aligned}$$

These formal calculations will be justified if two interchanges of integrations are justified and the conditions for the asymptotic evaluation are met.

For the first interchange of integrations we must show that

$$\int_0^{\infty} |\cos(2kx)| dx \int_0^{\infty} e^{-k\sigma u(x^2+1)} |u^{-1} f(u^{-1})| du < \infty.$$

But, by assumption (2) and Lemma 1, if $k\sigma > \gamma$ then the inner integral is $O(x^{-2m})$ as $x \rightarrow \infty$ and $m > \frac{1}{2}$. Thus the interchange is justified.

For the second interchange we must show that

$$\int_0^{\infty} e^{-k\sigma u} |u^{-3/2} f(u^{-1})| du \int_0^{\infty} e^{-v^2} |\cos\{2(k\sigma u)^{-1/2} v\}| dv$$

exists. But this is obvious since the inner integral is less than $\frac{1}{2}\sqrt{\pi}$ and since

$$\int_0^{\infty} e^{-k\sigma u} u^{-3/2} f(u^{-1}) du$$

converges absolutely by assumption (2) and Lemma 1.

The verification of the conditions for the asymptotic evaluation is exactly the same as in Theorem 1.

5. Representation theorems.

THEOREM 3. If $f(s)$ satisfies conditions (1) and (2) of Theorem 2, and

$$(3) \quad |L_{k,i}[f(s)]| < M, \quad k > k_0, 0 \leq t < \infty,$$

then there exists a $\phi(t)$, bounded in $0 \leq t < \infty$, and such that

$$f(s) = \int_0^\infty e^{-st} \phi(t) dt.$$

Proof. We shall use Widder [3, p. 33, theorem 17b]. By this theorem, there exists an increasing and unbounded sequence of numbers $\{k_i\}$, and a bounded function $\phi(t)$ such that

$$\lim_{t \rightarrow \infty} \int_0^\infty e^{-st} L_{k_i,i}[f(s)] dt = \int_0^\infty e^{-st} \phi(t) dt.$$

But, because of (3), $f(s)$ satisfies all the postulates of Theorem 2. Thus

$$\lim_{t \rightarrow \infty} \int_0^\infty e^{-st} L_{k_i,i}[f(s)] dt = f(\sigma),$$

so that

$$f(\sigma) = \int_0^\infty e^{-\sigma t} \phi(t) dt.$$

THEOREM 4. If $f(s)$ satisfies conditions (1) and (2) of Theorem 2, and

$$(3) \quad \int_0^\infty |L_{k,i}[f(s)]|^p dt < M^p, \quad k > k_0, p > 1,$$

then, there exists a $\phi(t) \in L_p(0, \infty)$ such that

$$f(s) = \int_0^\infty e^{-st} \phi(t) dt.$$

Proof. This theorem is proved in exactly the same manner as Theorem 3, but using Widder [3, p. 33, theorem 17a].

6. Inversion and representation theory for the Laplace-Stieltjes transform. In this section we shall regard $f(s)$ as defined by

$$\text{III} \quad f(s) = \int_0^\infty e^{-st} d\alpha(t)$$

THEOREM 5. If

- (1) $\alpha(t)$ is of bounded variation in $(0, T)$ for all $T > 0$, $\alpha(0) = \alpha(0+) = 0$,
- (2) $e^{-st} t^{-3/2} \alpha(t) \in L(0, \infty)$ for $s > \gamma$,

then

$$\lim_{k \rightarrow \infty} \int_0^t L_{k,t}[f(s)] dt = \frac{1}{2} \{a(t+) + a(t-)\}$$

almost everywhere for $t > 0$.

Proof. Clearly $f(s)$ exists for $s > \gamma$. To prove the assertion we shall use the same theorems of Widder used in Theorem 1.

$$f(s) = \int_0^\infty e^{-st} d\alpha(t) = s \int_0^\infty e^{-st} \alpha(t) dt.$$

Operating formally we have,

$$\begin{aligned} L_{k,t}[f(s)] &= (ke^{2k}(\pi t)^{-1}) \int_0^\infty x^{-1} \cos(2kx^{1/2})(k(x+1)/t) dx \int_0^\infty e^{-k(x+1)u/t} \alpha(u) du \\ &= (k^2 e^{2k}(\pi t^3)^{-1}) \int_0^\infty e^{-ku/t} \alpha(u) du \int_0^\infty e^{-kxu/t} x^{-1/2} (x+1) \cos(2kx^{1/2}) dx \\ &= (2k^{3/2} e^{2k}(\pi t^{3/2})^{-1}) \int_0^\infty e^{-ku/t} \alpha(u) u^{-1/2} du \int_0^\infty e^{-v^2} (kv^2(ku)^{-1} + 1) \cos\{2(ktu^{-1})^{1/2}v\} dv \\ &\quad \text{(where } v^2 = kxu/t) \\ &= (2k^{3/2} e^{2k}(\pi t^{3/2})^{-1}) \int_0^\infty e^{-k(u t^{-1} + tu^{-1})} \alpha(u) u^{-1/2} (1 + t(2ku)^{-1} - t^2 u^{-3}) du \\ &= e^{2k} (k/\pi)^{1/2} \int_0^\infty \frac{d}{dt} (e^{-k(u t^{-1} + tu^{-1})} t^{1/2} u^{-3/2} \alpha(u)) du. \end{aligned}$$

Thus

$$\int_0^t L_{k,t}[f(s)] dt = e^{2k} (k/\pi)^{1/2} \int_0^\infty e^{-k(u t^{-1} + tu^{-1})} u^{-3/2} \alpha(u) du \rightarrow \frac{1}{2} (a(t+) + a(t-))$$

as $k \rightarrow \infty$.

These formal calculations will be justified if the two interchanges of integrations are justified and the asymptotic evaluation is justified.

The first interchange of integrations is justified in almost the same manner as in Theorem 1, as is also the asymptotic evaluation.

For the second interchange of integrations, consider

$$\int_0^\infty e^{-k(u t^{-1} + tu^{-1})} u^{-3/2} \alpha(u) du.$$

By assumption (2), this integral converges uniformly and absolutely in any bounded closed interval for which $k/t > \gamma$.

Thus

$$\frac{d}{dt} \int_0^\infty e^{-k(u t^{-1} + tu^{-1})} \alpha(u) t^{1/2} u^{-3/2} du = \int_0^\infty \frac{d}{dt} \{e^{-k(u t^{-1} + tu^{-1})} \alpha(u) t^{1/2} u^{-3/2}\} du,$$

and the second interchange is justified.

THEOREM 6. If $f(s)$ satisfies conditions (1) and (2) of Theorem 2, and

$$(3) \quad \int_0^t |L_{k,t}[f(s)]| dt < M, \quad k > k_0, 0 < t < \infty,$$

then there exists $a(t)$, of bounded variation in $0 < t < \infty$ such that

$$f(s) = \int_0^\infty e^{-st} da(t).$$

Proof. This theorem is proved in exactly the same manner as Theorem 3, but using Widder [3, p. 31, theorem 16.4].

REFERENCES

1. A. Erdélyi, *The inversion of the Laplace transformation*, Math. Mag., vol. 29 (1950-51).
2. I. I. Hirschman Jr., *A new representation and inversion theory for the Laplace integral*, Duke Math. J., vol. 15 (1948).
3. D. V. Widder, *The Laplace transformation* (Princeton, 1941).

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A NOTE ON DIVERGENT SERIES

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1. Methods of summation of Rogosinski and Bernstein. In this note we shall discuss certain matrix methods of summation, though otherwise, §1 and §2 are not connected.

In this section, we shall study some properties of the method (B^h) where we say the series $\sum u_n$ is summable (B^h) when

$$B_n^h = \sum_{p=0}^n u_p \cos \frac{\pi}{2} \left(\frac{p}{n+h} \right) \rightarrow s, \quad n \rightarrow \infty.$$

The method (B^h) has been studied in special cases arising from different values of h by Rogosinski [11; 12], Bernstein [2], and more recently by Karamata [3; 4].

Two methods (A) and (B) are equivalent, $(A) \equiv (B)$, when all series summable (A) are summable (B) to the same sum and inversely; on the other hand, the method (B) is more powerful than the method (A) , $(A) \subset (B)$, when all series summable (A) are summable (B) to the same sum.

In the paper of Karamata [3] a theorem states that $(B^h) \equiv (C_1)$ if $0 < h < 1$, $|h - \frac{1}{2}| > .19$ where (C_1) denotes the Cesàro method. Lorentz [6] pointed out that his proof contains gaps, but can be made valid if $.69 < h < 1$. If $h = \frac{1}{2}$, then (B^h) is more powerful than (C_1) [4]. Here we shall prove Karamata's theorem for $\frac{1}{2} < h$; our proof will be simpler than that given in [3].

The partial sums B_n^h of the (B^h) method may be expressed, after easy calculations, in terms of σ_n , the partial sums of the (C_1) method. The transformation from σ_n to B_n^h is regular and hence any (C_1) -summable series is summable (B^h) for all h , i.e., $(C_1) \subset (B^h)$.

Our main theorem is

THEOREM 1.1 $(B^h) \equiv (C_1)$ for $h > \frac{1}{2}$.

In our proof we shall need a theorem of Agnew [1], which was rediscovered by Rado [10]. In the formulation of Rado, if the method (T) :

$$t_m = \sum_{n=0}^{\infty} c_{mn} s_n,$$

is regular and if $c_{mv} = 0$, $v > m$,

$$\sum_{n=0}^{m-1} |c_{nv}| < \theta |c_{mm}|, \quad \theta < 1$$

for almost all m , then (T) is equivalent to convergence.

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We shall introduce the method (K^c) where

$$K_n^c = \frac{1}{n+1} \sum_{s=0}^n [(1-c)s_{s-1} + cs_s] \rightarrow s, \quad n \rightarrow \infty.$$

If we express the partial sums of (K^c) in terms of σ_s , the partial sums of (C_1) ,

$$K_n^c = (1-c) \frac{n}{n+1} \sigma_{n-1} + c\sigma_n$$

it follows at once from the theorem of Agnew that $(K^c) = (C_1)$ if $c > \frac{1}{2}$. We shall now prove that $(B^h) = (K^h)$ for $h > \frac{1}{2}$ and the proof of Theorem 1.1 will then follow.

THEOREM 1.2 $(B^h) = (K^h)$ if $h > \frac{1}{2}$.

Proof. We have

$$B_n^h = \sum_{s=0}^n u_s \cos \frac{\pi}{2} \frac{s}{n+h},$$

$$K_n^c = \frac{1}{n+1} \sum_{s=0}^n [(1-c)s_{s-1} + cs_s].$$

Solving for s_s , we have

$$cs_s = (\nu+1)K_s^c - \frac{1}{c} \nu K_{s-1}^c + \frac{1-c}{c} (\nu-1)K_{s-2}^c + \dots + (-1)^s \left(\frac{1-c}{c} \right)^{\nu-1} \frac{1}{c} K_0^c$$

or

$$s_s = \left(1 - \frac{1}{c}\right)^s \sum_{\mu=0}^s \frac{c^{\mu-1}}{(1-c)^{\mu-1}} (-1)^\mu (\mu+1) K_\mu^c,$$

where the prime means that the term with $\mu = \nu$ has the additional factor $(1-c)$. Substituting in B_n^h , we obtain, with $\theta = \pi/2(n+h)$ and $a = 1 - 1/c$,

$$\begin{aligned} B_n^h &= \sum_{s=0}^n \cos \frac{\pi}{2} \frac{s}{n+h} (s_s - s_{s-1}) = \sum_{s=0}^{n-1} s_s \{\cos \mu\theta - \cos (\mu+1)\theta\} + s_n \cos n\theta \\ &= -\frac{1}{c} \sum_{\mu=0}^{n-1} \{\cos \mu\theta - \cos (\mu+1)\theta\} a^\mu \sum_{s=0}^s a^{-(s+1)} (\nu+1) K_s^c \\ &\quad - \frac{1}{c} \cos n\theta a^n \sum_{s=0}^n a^{-(s+1)} (\nu+1) K_s^c, \end{aligned}$$

and changing the order of summation in the first sum,

$$\begin{aligned} B_n^h &= -\frac{1}{c} \sum_{\mu=0}^{n-1} a^{-(\mu+1)} (\nu+1) K_\mu^c \left[\sum_{s=\mu}^{n-1} a^s \{\cos \mu\theta - \cos (\mu+1)\theta\} + a^n \cos n\theta \right] \\ &\quad + \frac{1}{c} (n+1) K_n^c \cos n\theta. \end{aligned}$$

Here, the expression in square brackets is

$$\begin{aligned} & (1-c)a^r\{\cos r\theta - \cos(r+1)\theta\} + a^{r+1}\{\cos(r+1)\theta - \cos(r+2)\theta\} \\ & \quad + \dots + a^{n-1}\{\cos(n-1)\theta - \cos n\theta\} + a^n \cos n\theta \\ & = -ca^r\{\cos r\theta - \cos(r+1)\theta\} + a^r \cos r\theta - \frac{1}{c}a^r \cos(r+1)\theta \\ & \quad + \dots - \frac{1}{c}a^{n-1} \cos n\theta. \end{aligned}$$

Using the formula

$$\begin{aligned} & a^{r+1} \cos(r+1)\theta + \dots + a^n \cos n\theta \\ & = \Re(a^{r+1}e^{i(r+1)\theta} + \dots + a^ne^{in\theta}) = \Re \frac{a^{r+1}e^{i(r+1)\theta} - a^{n+1}e^{i(n+1)\theta}}{1 - ae^{i\theta}} \\ & = \frac{a^{r+1} \cos(r+1)\theta - a^{n+1} \cos(n+1)\theta + a^{r+2} \cos r\theta - a^{n+2} \cos n\theta}{1 - 2a \cos \theta + a^2}, \end{aligned}$$

we obtain, for the above expression,

$$\begin{aligned} & [-ca^{r+1} \cos r\theta + ca^r \cos(r+1)\theta] \\ & \quad - \frac{a^r \cos(r+1)\theta - a \cos r\theta}{c} - \frac{a^n \cos n\theta - \cos(n+1)\theta}{c} \cdot \frac{a}{1 - 2a \cos \theta + a^2}, \end{aligned}$$

so we have

$$\begin{aligned} B_n^h = & -\frac{1}{c} \sum_{r=0}^{n-1} a^{-(r+1)}(r+1)K_r^c \left\{ -\frac{a^r}{c} \frac{a \cos n\theta - \cos(n+1)\theta}{1 - 2a \cos \theta + a^2} \right. \\ & \left. + \frac{4ca^{r+1} \sin^2 \frac{1}{2}\theta [\cos(r+1)\theta - a \cos r\theta]}{1 - 2a \cos \theta + a^2} \right\} + \frac{n+1}{c} K_n^c \cos n\theta. \end{aligned}$$

We shall now estimate the sum of the absolute values of the coefficients of K_r^c and show that the sum of the first $n-1$ of them is less than that of K_n^c . Under these conditions we apply the theorem of Agnew.

Here, for the coefficient of K_n ,

$$\lim_{n \rightarrow \infty} \frac{1}{c}(n+1) \cos n\theta = \frac{\pi}{2} \cdot \frac{h}{c}.$$

We break the sum of the absolute values of the other coefficients into two parts, the second part of which is

$$\begin{aligned} D_2 = & \sum_{r=0}^{n-1} \left| -\frac{1}{c^2} \frac{1}{a^{r+1}}(r+1) \frac{4ca^{r+1} \sin^2 \frac{1}{2}\theta [\cos(r+1)\theta - a \cos r\theta]}{1 - 2a \cos \theta + a^2} \right| \\ & = \left| \frac{1}{c} \frac{4 \sin^2 \frac{1}{2}\theta}{1 - 2a \cos \theta + a^2} \right| \sum_{r=0}^{n-1} (r+1) |\cos(r+1)\theta - a \cos r\theta|. \end{aligned}$$

Since we have

$$|\cos(\nu+1)\theta - a \cos \nu\theta| \leq |\cos(\nu+1)\theta - \cos \nu\theta| + \left|\frac{1}{c}\right| |\cos \nu\theta|,$$

$$\sum_{\nu=0}^{n-1} (\nu+1) |\cos(\nu+1)\theta - a \cos \nu\theta| \leq A(n+1) + \left|\frac{1}{c}\right| \frac{n(n+1)}{2},$$

$$\sin^2 \frac{\theta}{2} = \frac{1}{n^2} \left\{ \frac{\pi^2}{16} + o(1) \right\},$$

and $1 - 2a \cos \theta + a^2 = (1-a)^2 + o(1) = c^{-2} + o(1)$, therefore

$$D_2 \leq \frac{\pi^2}{8} + o(1).$$

Now we shall turn our attention to the first part of the sum

$$D_1 \leq \left| \frac{1}{c^3} \right| \left| \frac{a^n \cos n\theta - a^{n-1} \cos(n+1)\theta}{1 - 2a \cos \theta + a^2} \right| \sum_{\nu=0}^{n-1} (\nu+1) |a^{-\nu}|.$$

As before $1 - 2a \cos \theta + a^2 = c^{-2} + o(1/n)$, and therefore

$$\begin{aligned} D_1 &\leq \left| \frac{a^{n-1}}{c} \right| |a \cos n\theta - \cos(n+1)\theta| [1 + o(1/n)]^{-1} \frac{(n+1) |a|^{-n} + n |a|^{-n-1} + 1}{(1 - 1/|a|)^2} \\ &\leq \left| \frac{1}{c} \right| |a \cos n\theta - \cos(n+1)\theta| \frac{n(1 - |a|) + o(1)}{(1 - |a|)^2}. \end{aligned}$$

Here we have assumed that $a^n = o(1)$, that is, $|a| < 1$. We shall proceed to give an estimate of $|a \cos n\theta - \cos(n+1)\theta|$. We have

$$\begin{aligned} |a \cos n\theta - \cos(n+1)\theta| &= \left| a \sin \frac{\pi}{2} \frac{h}{n+h} - \sin \frac{\pi}{2} \frac{h-1}{n+h} \right| \\ &= \left| \left(a - \frac{h-1}{h} \right) \sin \frac{\pi}{2} \frac{h}{n+h} + \frac{h-1}{h} \sin \frac{\pi}{2} \frac{h}{n+h} - \sin \frac{\pi}{2} \frac{h-1}{n+h} \right| \\ &\quad \frac{h-1}{h} \sin \frac{\pi}{2} \frac{h}{n+h} - \sin \frac{\pi}{2} \frac{h-1}{n+h} = o\left(\frac{1}{n^2}\right), \end{aligned}$$

and so

$$\begin{aligned} |a \cos n\theta - \cos(n+1)\theta| &\leq \left| \left(a - \frac{h-1}{h} \right) \sin \frac{\pi}{2} \frac{h}{n+h} + o\left(\frac{1}{n^2}\right) \right| \\ &\leq \frac{\pi}{2n} \left| 1 - \frac{h}{c} \right| + o\left(\frac{1}{n}\right). \end{aligned}$$

Substituting the above estimate for $|a \cos n\theta - \cos(n+1)\theta|$ in our expression for D , we obtain

$$D_1 \leq \frac{\pi}{2c} \left| 1 - \frac{h}{c} \right| \frac{1}{1 - |a|}.$$

To satisfy the theorem of Agnew, the absolute value of the coefficient of

K_n^c must be greater than the sum of the absolute values of the other coefficients. In our case, this is true if

$$(1.1) \quad \frac{\pi|h|}{2|c|} > \frac{\pi^2}{8} + \frac{\pi}{2c} \left| 1 - \frac{h}{c} \right| \frac{1}{|1 - |a||}.$$

If $c = h$, this reduces to

$$\frac{\pi}{2} > \frac{\pi^2}{8},$$

so that $(B^h) = (K^h)$ whenever $|a| < 1$ or $h > \frac{1}{2}$. This completes our proof.

In the general case, (1.1) does not hold for $h < \frac{1}{2}$ while $c > \frac{1}{2}$; so that (B^h) , $h < \frac{1}{2}$ can not be shown equivalent to some (K^c) , $c > \frac{1}{2}$ by these means. Examples can be constructed to show (B^h) is not equivalent to (C_1) for $h < 0$. The most interesting question remaining open is whether or not (B^h) is equivalent to (C_1) in the interval $0 < h < \frac{1}{2}$.

2. Some special Nörlund methods of summation. In this section we wish to consider some elementary Nörlund methods, namely, methods of the form

$$(A) \quad \sigma_n = a_0 s_{n-p} + \dots + a_p s_n, \quad a_0 + a_1 + \dots + a_p = 1.$$

It was first proposed as a problem by Pólya, [9] that the method defined by

$$t_n = (1 - c) s_{n-1} + c s_n \rightarrow s, \quad n \rightarrow \infty \quad (c \neq 0),$$

is equivalent to convergence if and only if $c > \frac{1}{2}$. Kubota [5] proved more generally that a transformation of type (A) is equivalent to convergence if and only if all of the roots of the "associated" equation

$$(2.1) \quad a_0 + a_1 z + \dots + a_p z^p = 0,$$

lie inside the unit circle.

Other results concerning the method (A) have been obtained by Lorentz [7] and by Silverman and Szász [13]. We shall show that any bounded sequence summable (A) is convergent if and only if none of the roots of (2.1) lie on the unit circle. This will easily follow from Theorem 2.2 (the main theorem of this section), where we describe all (A)-summable sequences under the above hypotheses on the roots of (2.1).

We shall first prove

LEMMA 1. If

$$s_n = \sum_{r=1}^n r^k a^r \quad (a \neq 1)$$

then S_n may be written in the form $P_k(n) a^n + c$ where $P_k(n)$ is of the form

$$c_k n^k + c_{k-1} n^{k-1} + \dots + c_0,$$

and $c_k, c_{k-1}, \dots, c_0, c$ are constants depending only on a .

Let us write

$$\Delta^k f(v) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i} f(v+i).$$

Applying Abel's formula, we have

$$\sum_{v=1}^n v^k a^v = \frac{1}{1-a} - \frac{a}{1-a} \sum_{v=1}^n a^v \Delta v^k - \frac{a^n}{1-a} (n+1)^k.$$

Repeating this process $k+1$ times,

$$\begin{aligned} \sum_{v=1}^n v^k a^v = & \left\{ \frac{1}{1-a} + \frac{a}{(1-a)^2} \Delta 1^k + \dots + (-1)^k \frac{a^k}{(1-a)^k} \Delta^k (n+1)^k \right\} \\ & + (-1)^k \frac{a^k}{(1-a)^k} \sum_{v=1}^n a^v \Delta^{k+1} v^k - \frac{a^n}{1-a} \left\{ (n+1)^k - \frac{a}{1-a} \Delta (n+1)^k + \right. \\ & \left. \dots + (-1)^k \frac{a^k}{(1-a)^k} \Delta^k (n+1)^k \right\} \end{aligned}$$

and therefore, since $\Delta^{k+1} v^k = 0$,

$$\sum_{v=1}^n v^k a^v = P_k(n) a^n + c$$

as required.

In preparation for Theorem 2.2 we shall first consider the special case of (A) when $p = 1$. In this case, we may write (A) in the form

$$(A_s) \quad \sigma_n = \frac{1}{1-a} \{-as_{n-1} + s_n\} \quad (a \neq 1).$$

THEOREM 2.1 Suppose $|a| > 1$.

(i) If $\sigma_n \rightarrow \sigma$, then $s_n = ca^n + \sigma'_n$, where $\sigma'_n \rightarrow \sigma$ and c is a certain constant.

(ii) If $\sigma_n = P_\mu(n)a^n$ where $P_\mu(n) = c_\mu n^\mu + c_{\mu-1}n^{\mu-1} + \dots + c_0$, then

$$s_n = (c'_{\mu+1}n^{\mu+1} + c'_\mu n^\mu + \dots + c'_0)a^n = P'_{\mu+1}(n)a^n$$

and conversely.

(iii) If $\sigma_n = P_r(n)b^n$; $P_r(n) = c'r^r + c_{r-1}n^{r-1} + \dots + c_0$ and $b \neq a$ then

$$s_n = (c'_r n^r + \dots + c'_0) b^n + ca^n$$

and conversely.

Proof. We have for (i),

$$(2.2) \quad \frac{1}{1-a} s_n = a^n \left[\sigma_0 + \frac{\sigma_1}{a} + \dots + \frac{\sigma_n}{a^n} \right].$$

If we define $t_n = \sigma_n/a^n$, then part (i) of our theorem means that, for $|a| > 1$, $a^n t_n \rightarrow \sigma$ implies $t_0 + t_1 + \dots + t_n = c + \sigma'_n/a^n$, where $(1-a)\sigma'_n \rightarrow \sigma$.

The series $\sum t_n$ is absolutely convergent. Set

$$c = \sum_{n=0}^{\infty} t_n,$$

then

$$\begin{aligned} t_0 + t_1 + \dots + t_n &= c - (t_{n+1} + t_{n+2} + \dots) \\ &= c - \frac{1}{a^n} \left[\frac{1}{a} a^{n+1} t_{n+1} + \frac{1}{a^2} a^{n+2} t_{n+2} + \dots \right]. \end{aligned}$$

Since $a^n t_n \rightarrow \sigma$,

$$\sum_{k=1}^{\infty} \frac{1}{a^k} a^{n+k} t_{n+k}$$

converges toward

$$\sigma \left(\frac{1}{a} + \frac{1}{a^2} + \dots + \frac{1}{a^n} + \dots \right) = \sigma \frac{1}{a} \frac{1}{1 - 1/a} = \frac{\sigma}{a - 1}.$$

Therefore

$$t_0 + \dots + t_n = c + \frac{1}{a^n} \sigma'_n, \quad \sigma'_n \rightarrow \frac{\sigma}{a - 1},$$

which proves (i).

(ii) Substituting the value of σ_n in (2.2) we have

$$\begin{aligned} \frac{1}{1-a} s_n &= a^n \{ c_n (1^n + 2^n + \dots + n^n) + c_{n-1} (1^{n-1} + 2^{n-1} + \dots + n^{n-1}) \\ &\quad + \dots + c_0 (1 + 1 + \dots + 1) \}. \end{aligned}$$

Using the well-known fact that $1^n + 2^n + \dots + n^n$ is a polynomial in n of degree $\mu + 1$ with constant coefficients, we obtain

$$s_n = \{ c'_{\mu+1} n^{\mu+1} + c'_\mu n^\mu + \dots + c'_0 \} a^n = P'_{\mu+1}(n) a^n.$$

The converse becomes evident on substituting the expression for s_n in (A₀).

(iii) Again we substitute the value for σ_n in (2.2),

$$\begin{aligned} \frac{1}{1-a} s_n &= a^n \left\{ c_n \left(1^n \frac{b}{a} + 2^n \frac{b^2}{a^2} + \dots + n^n \frac{b^n}{a^n} \right) \right. \\ &\quad \left. + c_{n-1} \left(1^{n-1} \frac{b}{a} + 2^{n-1} \frac{b^2}{a^2} + \dots + n^{n-1} \frac{b^n}{a^n} \right) + \dots + c_0 \left(1 + \frac{b}{a} + \dots + \frac{b^n}{a^n} \right) \right\}. \end{aligned}$$

By Lemma 1,

$$(2.3) \quad 1^n \frac{b}{a} + 2^n \frac{b^2}{a^2} + \dots + n^n \frac{b^n}{a^n} = \frac{b^n}{a^n} P'_\mu(n) + c,$$

where $P'_\mu(n) = (C'_\mu n^\mu + \dots + c'_0)$.

Using (2.3) we have

$$\frac{1}{1-a} s_n = a^n \left\{ c_n P'_\mu(n) \frac{b^n}{a^n} + c_{n-1} + \dots + c_1 P'_1(n) \frac{b^n}{a^n} + c_{11} + c_0 \frac{b^n}{a^n} + c_{01} \right\}$$

which may be written $P'v(n)b^n + ca^n$. Again the converse is evident if we substitute s_n in (A_s) .

We now return to the method (A).

THEOREM 2.2 *If $a_1, a_2, \dots, a_k, |a_i| \neq 1$ are all of the different roots of equation (2.1), a_1, a_2, \dots, a_k are those roots with $|a_i| > 1$ and m_1, m_2, \dots, m_k their multiplicities, then the general form of a sequence summable (A) is*

$$(2.4) \quad s_n = P_1(n)a_1^n + P_2(n)a_2^n + \dots + P_k(n)a_k^n + s'_n,$$

where

$$P_i(n) = c_{i,m_i-1}n^{m_i-1} + c_{i,m_i-2}n^{m_i-2} + \dots + c_{i,0}$$

are polynomials in n of degree m_{i-1} with arbitrary constant coefficients and s'_n is an arbitrary convergent sequence.

Proof. (A) may be considered as an iteration of p transformations

$$\sigma_n^{j-1} = \frac{1}{1-b_j} \{-b_j \sigma_{n-1}^j + \sigma_n^j\}, \quad j = 1, 2, \dots, p, \sigma_n^0 = \sigma_n, \sigma_n^p = s_n.$$

The b_j are first those a_i with $|a_i| < 1$ and then the a_1, a_2, \dots, a_k all taken with their multiplicities. There will be m_1 transformations with $b_j = a_1$ and so on. The first $m = m_{k+1} + \dots + m_1$ transformations are all equivalent to convergence by the theorem of Kubota, and therefore the convergence of σ_n will be equivalent to the convergence of σ_n^m .

Hence, in proving our theorem we may assume that all $|a_i| > 1$. For the first transformation σ_n^0 is a convergent sequence, and therefore

$$\sigma'_n = ca_1^n + \bar{\sigma}'_n, \quad \bar{\sigma}'_n \rightarrow \sigma$$

by Theorem 2.1 (i). If now we repeat this argument p times and use Theorem 2.1 (i), (ii), and (iii), we shall obtain as the final result expression (2.4) for s_n . Conversely, substituting s_n in the expression for (A), we see s_n is (A) summable. This proves the theorem.

We shall next prove a lemma that will enable us to prove a further theorem on methods of type (A).

LEMMA 2. *If $|a_i| > 1$ for $a_i \neq a_j, i \neq j$ ($i = 1, 2, \dots, k$), and*

$$P_{\mu_i}(n) = c_{\mu_i} n^{\mu_i} + \dots + c_{i0}, \quad P_{\mu_i}(n) \neq 0 \text{ for all } i,$$

then the expression

$$(2.5) \quad y_n = P_{\mu_1}(n)a_1^n + P_{\mu_2}(n)a_2^n + \dots + P_{\mu_k}(n)a_k^n,$$

is unbounded for $n \rightarrow \infty$.

We shall show that if $y_n = O(1)$ we have a contradiction. Assume the first l of the a_i are all those having that modulus which is the maximum modulus of the a_i that is

$$|a_1| = |a_2| = \dots = |a_l|,$$

and

$$a_i = a_1 e^{ia_i}, \quad a_1 = 0, \quad a_i \neq a_j, \quad i \neq j \quad (i = 1, 2, \dots, l).$$

Then (2.5) becomes

$$y_n = a_1^n [P_{\mu_1}(n) + e^{ia_2} P_{\mu_2}(n) + \dots + e^{ia_l} P_{\mu_l}(n)] + a_1^n o(1).$$

We have, since $y_n = O(1)$,

$$(2.6) \quad P_{\mu_1}(n) + e^{ia_2} P_{\mu_2}(n) + \dots + e^{ia_l} P_{\mu_l}(n) = o(1).$$

We write C_{μ_i} for the coefficients in

$$P_{\mu_i}(n) \text{ of } n^\mu, \quad \mu = \max_{i=1, 2, \dots, l} \mu_i;$$

at least one of these is different from zero. We consider the l equations

$$c'_{\mu_1} + e^{i(n+j)a_2} c'_{\mu_2} + \dots + e^{i(n+j)a_l} c'_{\mu_l} = \epsilon_{n+j}, \quad j = 0, 1, 2, \dots, l-1.$$

Dividing by n^μ in (2.6) we have $\epsilon_{n+j} \rightarrow 0$, $n \rightarrow \infty$. The a_j are all different and different from zero. Solving these l equations

$$(2.7) \quad c'_{\mu_i} e^{ia_i n} = \begin{vmatrix} 1 & 1 & \dots & 1 & \epsilon_n & \dots & 1 \\ 1 & e^{ia_2} & \dots & e^{ia_{l-1}} & \epsilon_{n+1} & \dots & e^{ia_l} \\ & & \dots & & & \dots & \\ 1 & e^{ia_2(l-1)} & \dots & e^{ia_{l-1}(l-1)} & \epsilon_{n+l-1} & \dots & e^{ia_l(l-1)} \end{vmatrix} V^{-1},$$

where V is a Vandermonde determinant different from zero and independent of n .

Hence, expanding the numerator in (2.7) by the j th column, we see that $c'_{\mu_i} e^{ia_i n} \rightarrow 0$ as $n \rightarrow \infty$ or $c'_{\mu_i} = 0$ for all j . This contradiction proves our lemma.

THEOREM 2.3 Any bounded sequence summable (A) is convergent if and only if none of the roots of (2.1) lie on the unit circle.

Proof. The sufficiency of these conditions follows from Theorem 2.2 and Lemma 2.

If we assume that the associated equation (2.1) has a root a with $|a| = 1$, then breaking (A) into an iteration of transformations as in Theorem 2.2, we can consider

$$(2.8) \quad \sigma_n^{p-1} = \frac{1}{1-a} [-as_{n-1} + s_n],$$

last in our sequence of transformations. It is then evident that the method (2.8) and therefore (A) sums the sequence $e^{in\psi}$ where $a = e^{i\psi}$. This contradiction proves our conditions necessary.

The existence of a bounded divergent (A)-summable sequence implies [8] that sequences of his type form a non-separable subset of the space m of bounded sequences. It follows that in the case of a root $|a| = 1$ a simple enumeration of all (A)-summable sequences comparable with (2.4) is impossible.

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REFERENCES

1. R. P. Agnew, *On equivalence of methods of evaluation of sequences*, Tôhoku Math. J., vol. 35 (1932), 244-252.
2. S. N. Bernstein, *Sur un procédé de sommation des series trigonometriques*, C. R. Acad. Sci., Paris, vol. 191 (1930), 976.
3. J. Karamata, *Sur la sommabilité de S. Bernstein et quelques procédés de sommation qui s'y rattachent*, Mat. Sborn., N. S. 21 (63) (1947), 13-24.
4. ———, *Über die Beziehung zwischen dem Bernsteinschen und Cesàroschen Limitierungsverfahren*, Math. Z., vol. 52 (1949), 305.
5. T. Kubota, *Ein Satz über den Grenzwert*, Tôhoku Math. J., vol. 12 (1917), 222-224.
6. G. G. Lorentz, *Review of Karamata [3]*, Zentralblatt für Math., vol. 29 (1948), 208.
7. ———, *A contribution to the theory of divergent sequences*, Acta Math., vol. 80 (1948), 167-190.
8. S. Mazur and W. Orlicz, *Sur les methodes linéaires de sommation*, C. R. Acad. Sci., Paris, vol. 196 (1933), 32.34.
9. G. Pólya, *Problem 509*, Archiv der Mathematik und Physik (3), vol. 24 (1915), 282.
10. R. Rado, *Some elementary Tauberian theorems I*, Quarterly J. Math. (Oxford ser.), vol. 9 (1938), 274-282.
11. W. Rogosinski, *Reihen Summierung durch Abschnittskoppelungen*, Math. Z., vol. 25 (1926), 132-149.
12. ———, *Über die Abschnitte Trigonometrischen Reihen*, Math. Ann., vol. 95 (1926), 110-124.
13. L. L. Silverman and O. Szász, *On a class of Nörlund matrices*, Ann. Math., vol. 45 (1944), 347-357.

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MULTIPLY SUBADDITIVE FUNCTIONS

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1. Introduction. Let S denote a Boolean ring with elements e , that is, a distributive, relatively complemented lattice with zero element 0 [2, p. 153]. In this paper we study real-valued functions $\Phi(e)$, $e \in S$ which have a representation of the form

$$(1.1) \quad \Phi(e) = \sup_{\phi \in C} \phi(e),$$

C being a certain class of *additive* functions on S ; $\phi(e)$ is additive if $\phi(e_1 \cup e_2) = \phi(e_1) + \phi(e_2)$ for any pair $e_1, e_2 \in S$, $e_1 \cap e_2 = 0$. We find a relation between (a) the possibility of representation (1.1); (b) the possibility of extension of $\Phi(e)$ onto a vector space X containing S ; (c) some simple intrinsic properties of $\Phi(e)$. For instance, one of our results (Theorem 4 in §5) is that $\Phi(e)$ possesses a representation (1.1), C being a family of additive and positive functions $\phi(e)$, if and only if $\Phi(e)$ is increasing and has the property

$$(1.2) \quad p\Phi(e) \leq \sum_{r=1}^n \Phi(e_r)$$

whenever the e_r cover e exactly p times (for a precise definition, see §§2,3). Functions Φ , satisfying (1.2), we call *multiply subadditive*; this property is stronger than the ordinary subadditivity expressed by the inequality

$$\Phi(e_1 \cup e_2) \leq \Phi(e_1) + \Phi(e_2), \quad e_1 \cap e_2 = 0.$$

On the other hand, we shall see that (1.2), with $=$ instead of \leq , holds for any additive function $\Phi(e)$. Multiply subadditive functions constitute, therefore, an intermediary class between the subadditive and the additive functions.

The problems treated in this paper arose, in the case when S is a Boolean ring of measurable sets, in connection with the study of certain spaces of functions, see [5, §4].

2. The vector space $X(S)$. A natural extension of a Boolean ring S into a space $X(S) = X = \{x\}$ is obtained as follows. Let x be any finite sum

$$x = \sum_{r=1}^n a_r e_r,$$

the order of terms being by definition irrelevant, where a_r are arbitrary real numbers and e_r arbitrary elements of S (with repetitions allowed). We define an equivalence relation $x \equiv y$ for two sums $x = \sum a_r e_r$, $y = \sum b_s f_s$ of this kind to mean that x can be transformed into y by a finite number of changes of the

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types: (A) a term ae in the sum x is replaced by $ae' + ae''$, if $e = e' \cup e''$, $e' \cap e'' = 0$, or conversely, $ae' + ae''$ is replaced by ae ; (B) $0e$ is omitted, or conversely, is added; (C) ae is replaced by $a'e + a''e$, where $a = a' + a''$, or conversely.

This equivalence relation is reflexive, symmetric, and transitive. Let X be the set of all equivalence classes and let X be provided with operations of addition and scalar multiplication as follows. If $x = \sum a_e e$, $y = \sum b_{f_\mu} f_\mu$, then

$$ax = \sum aa_e e, \quad x + y = \sum a_e e + \sum b_{f_\mu} f_\mu.$$

Clearly $x \equiv x_1$ and $y \equiv y_1$ imply $ax \equiv ax_1$ and $x + y \equiv x_1 + y_1$ and it follows that X is a vector space with zero of S as zero element.

The following lemma will be useful:

(2.1) *Two sums $x = \sum a_e e$, and $y = \sum b_{f_\mu} f_\mu$ are equivalent if and only if there are disjoint elements g_1, \dots, g_N such that every e , and every f_μ is a union of some of these g_ν in such a way that, if the e , f_μ are replaced by the sums of the corresponding g_ν and the terms reduced, the two expressions $\sum a_e e$, $\sum b_{f_\mu} f_\mu$ become identical.*

For the proof, we write $x \sim y$, if there are such g_ν . Clearly, $x \sim y$ implies $x \equiv y$. But also the converse is true. First, we have $x \sim x$ for any x . For if e_1, \dots, e_n is a finite set of elements of S , the e , can be expressed as unions of suitable disjoint g_ν . Such g_ν are obtained by taking all possible intersections $\bigcap_{\nu=1}^n e'_\nu$, where each e'_ν is either e_ν or the complement of e_ν with respect to $\bigcup_{\nu=1}^n e_\nu$. Again, the relation $x \sim y$ is not destroyed when any of the admissible changes (A), (B), (C) is performed on x . This shows that $x \sim y$ is equivalent to $x \equiv y$ and proves our assertion. In particular, it follows that if $\sum e_\nu$ and $\sum f_\mu$ are equivalent then $\bigcup e_\nu = \bigcup f_\mu$. As a corollary we obtain that two elements e_1, e_2 of S which are equivalent, are identical in S .

We can now describe the relation $pe = \sum_{\nu=1}^n e_\nu$ in X , that is, the equivalence

$$\sum_{\mu=1}^p f_\mu = \sum_{\nu=1}^n e_\nu,$$

where $f_1 = \dots = f_p = e$, more directly in terms of S . Using the g_1, \dots, g_N of (2.1) it follows that

(2.2) $pe = \sum e_\nu$ holds if and only if there are disjoint decompositions $e_\nu = \bigcup_{\mu=1}^p e_{\nu\mu}$ such that $e = \bigcup_{\mu=1}^p e_{\nu\mu}$ as a disjoint decomposition for every $\mu = 1, \dots, p$.

For instance, we may by induction on p define the decompositions $e_\nu = \bigcup e_{\nu\mu}$ as follows: let $e_{\nu\mu}$ be the union of those g_ν which satisfy $g_\nu \subset e_\nu$ and $g_\nu \subset e_\sigma$ for precisely $\mu - 1$ indices $\sigma < \nu$. If (2.2) holds, we shall say that the e_1, \dots, e_n cover e exactly p times. In the same way, we shall say that e_1, \dots, e_n cover e at least p times if there are disjoint decompositions $e_\nu = \bigcup_{\mu=1}^p e_{\nu\mu}$ with $e \subset \bigcup_{\nu=1}^n e_{\nu\mu}$ ($\mu = 1, \dots, p$). It is clear that this is the case if and only if there are $e'_\nu \subset e_\nu$ ($\nu = 1, \dots, n$) which cover e exactly p times.

We shall write $x \leq y$, $x, y \in X$ if there exist representations

$$x = \sum_1^n a_v e_v, \quad y = \sum_1^n b_v e_v,$$

with $a_v \leq b_v$ ($v = 1, \dots, n$). This relation is transitive by (2.1). For instance, $e_1 \leq e_2$ implies $e_1 \leq e_2$ in X .

3. Multiply subadditive functions. As stated in §1, a function $\Phi(e)$, $e \in S$ is multiply subadditive if $p\Phi(e) \leq \sum \Phi(e_v)$ whenever $pe = \sum e_v$ in X , that is, whenever the e_v cover e exactly p times. If $\Phi(e)$ is, moreover, increasing, $\Phi(e) \leq \Phi(e')$ for $e \leq e'$, then the last inequality holds even if the e_v cover e at least p times.

Writing $0 = 0 + 0$, $2 \cdot 0 = 0$, we obtain $\Phi(0) \leq 2\Phi(0)$, $2\Phi(0) \leq \Phi(0)$. Therefore, a multiply subadditive function has the property $\Phi(0) = 0$. If, in addition, Φ is increasing, it follows that $\Phi(e) \geq 0$, $e \in S$.

If $\Phi(e)$ is additive on S , we obtain an extension $F(x)$ of ϕ onto X by putting $F(x) = \sum a_v \phi(e_v)$ if $x = \sum a_v e_v$. Since the first sum is invariant under changes (A), (B), (C) of §2, $F(x)$ is a function defined on X . Clearly $F(x)$ is additive. In particular, we obtain

$$(3.1) \quad p\phi(e) = \sum a_v \phi(e_v), \quad pe = \sum a_v e_v,$$

so that any additive function ϕ on S is multiply subadditive with equality in (1.1). If, in addition, ϕ is positive, $\phi(e) \geq 0$, $e \in S$, then

$$(3.2) \quad \sum a_v \phi(e_v) \leq \sum b_v \phi(e_v), \quad \sum a_v e_v \leq \sum b_v e_v.$$

We finally remark that the condition

$$(3.3) \quad \Phi(e) \leq \sum_{v=1}^n a_v \Phi(e_v) \quad \text{whenever } e = \sum a_v e_v, a_v \geq 0,$$

is equivalent to multiple subadditivity. If the a_v are all rational, we write $a_v = k_v/k$ with positive integers k_v, k , and repeating each e_v exactly k_v times, deduce (3.3) from (1.2). In the general case we see, using (2.1), that, for fixed e_v, e , the relation $e = \sum a_v e_v$ is equivalent to a system of linear equations, with integral coefficients, for the a_v . Solutions a_1, \dots, a_n of this system can be approximated by positive rational solutions $a_1^{(m)}, \dots, a_n^{(m)}$. Then $a_v^{(m)} \rightarrow a_v$ for $m \rightarrow \infty$ and $e = \sum a_v^{(m)} e_v$. Making $m \rightarrow \infty$ in

$$\Phi(e) \leq \sum a_v^{(m)} \Phi(e_v),$$

we obtain (3.3).

4. Extension of functions from S onto X . In this section we connect the possibility of representation of the form

$$(4.1) \quad \Phi(e) = \sup_{\phi \in C} \phi(e),$$

$\phi(e)$ additive, with the possibility of extension of $\Phi(e)$ onto $X(S)$.

THEOREM 1. $\Phi(e)$ has a representation

$$(4.2) \quad \Phi(e) = \sup_{\phi \in C} |\phi(e)|$$

if and only if $\Phi(e)$ has an extension $P(x)$ onto X which satisfies the conditions

- (i) $P(x + y) \leq P(x) + P(y)$,
- (ii) $P(ax) = aP(x), \quad a > 0$,
- (iii) $P(x) \geq 0$,
- (iv) $P(-x) = P(x)$.

Proof. If (4.2) holds, we define

$$(4.3) \quad P(x) = \sup_{\phi \in C} |\sum a_r \phi(e_r)|, \quad x = \sum a_r e_r,$$

the value of $\sum a_r \phi(e_r)$ being independent of the choice of the representation $x = \sum a_r e_r$. Then $P(x)$ is finite, since

$$0 \leq P(x) \leq \sum |a_r| \Phi(e_r) < +\infty.$$

Also, $P(x)$ satisfies conditions (i)–(iv). Moreover, $P(x) = \Phi(e)$ for $x = e \in S$.

If, on the other hand, $\Phi(e)$ has an extension $P(x)$ of the required kind, we apply the Hahn-Banach theorem [1] and obtain, for each $e_0 \in S$, a linear functional $F(x)$ on X satisfying $F(e_0) = P(e_0) = \Phi(e_0)$ and $F(x) \leq P(x)$, $x \in X$. Then $F(x) \geq -P(-x) = -P(x)$, that is, $|F(x)| \leq P(x)$, $x \in X$. If C is the class of all functions $\phi(e) = F(e)$, $e \in S$ for all $F(x)$ of this kind, then (4.1) holds.

THEOREM 2. $\Phi(e)$ has a representation

$$(4.4) \quad \Phi(e) = \sup_{\phi \in C} \phi(e), \quad \phi(e) \geq 0,$$

where C is a class of positive additive functions ϕ if and only if $\Phi(e)$ has an extension $P(x)$ onto X with properties (i)–(iv) and

$$(v) \quad P(e_1) \leq P(e_2), \quad e_1 \subset e_2.$$

Proof. If $\Phi(e)$ satisfies (4.4), then $P(x)$, defined by (4.3), has the properties (i)–(v), so that they are necessary.

On the other hand, if $\Phi(e)$ has a continuation $P(x)$, then the proof of Theorem 1 establishes (4.4) where, however, the functions $\phi \in C$ are not necessarily positive. Let

$$\phi_1(e) = \sup_{e' \subset e} \phi(e') \geq 0$$

be the positive variation of $\phi \in C$. It is easy to see that ϕ_1 is additive and moreover (since $\Phi(e)$ increases by (v))

$$\Phi(e) = \sup_{e' \subset e} \Phi(e') = \sup_{\phi \in C} [\sup_{e' \subset e} \phi(e')] = \sup_{\phi_1 \in C_1} \phi_1(e),$$

which establishes (4.4) with $C_1 = \{\phi_1\}$ instead of C .

5. Representation of multiply subadditive functions. In this section we give the main results of this paper which connect the possibility of representation of a function $\Phi(e)$ in the form $\Phi(e) = \sup \phi(e)$ with the multiple subadditivity of $\Phi(e)$.

THEOREM 3. *A function $\Phi(e)$ on S has a representation*

$$(5.1) \quad \Phi(e) = \sup_{\phi \in C} |\phi(e)|$$

if and only if $\Phi(e)$ satisfies the condition

$$(5.2) \quad \Phi(e) \leq \sum_{r=1}^n |a_r| \Phi(e_r) \quad \text{whenever } e = \sum a_r e_r.$$

Proof. We begin by remarking that (5.1) and (5.2) both imply $\Phi(e) \geq 0$, the latter condition by putting $e = e - e + e$. If (5.1) holds and $e = \sum a_r e_r$, then

$$|\phi(e)| = |\sum a_r \phi(e_r)| \leq \sum |a_r| \Phi(e_r)$$

and (5.2) follows. Conversely, if this condition is fulfilled, we set

$$(5.3) \quad P(x) = \inf \sum |a_r| \Phi(e_r)$$

where the infimum is taken for all representations $x = \sum a_r e_r$. Then $0 \leq P(x) < +\infty$ and, by (5.2), $P(e) = \Phi(e)$, $e \in S$. As $P(x)$ satisfies (i)–(iv), we obtain (5.1) by Theorem 1.

Remark. As in the proof of (3.3), we may show that (5.2) is equivalent to the condition

$$(5.4) \quad p\Phi(e) \leq \sum_{r=1}^n \Phi(e_r) \quad \text{whenever } pe = \sum \pm e_r.$$

THEOREM 4. *A function $\Phi(e)$ on S admits a representation*

$$(5.5) \quad \Phi(e) = \sup_{\phi \in C} \phi(e), \quad \phi(e) \geq 0,$$

if and only if $\Phi(e)$ is increasing and multiply subadditive.

Proof. The necessity of the conditions is obvious. Conversely, let $\Phi(e)$ be increasing and multiply subadditive, we show that (5.2) holds. By the Remark, it is sufficient to prove (5.4). But if $pe = \sum \pm e_r$, then the e_r cover e at least p times (see §2) and therefore, by §3, we obtain (5.4) for the function $\Phi(e)$. As in Theorem 3, (5.3) gives an extension of $\Phi(e)$ onto X satisfying (i)–(iv). Also (v) is satisfied; hence our result follows from Theorem 2.

6. Special classes of multiply subadditive functions. Examples of multiply subadditive functions may be obtained by considering

$$(6.1) \quad \Phi(e) = F(\psi(e)),$$

where $\psi(e)$ is a fixed positive additive function on S and $F(u)$ a function of the real variable $u \geq 0$.

We shall assume that S is ψ -nonatomic, that is, if $\psi(e) = \delta$ for some $e \in S$ and $0 < \delta_1 < \delta$, there is an $e_1 \in e$ with $\psi(e_1) = \delta_1$. Clearly, with this condition, Φ is increasing if and only if F is increasing. Moreover, we have

THEOREM 5. A function (6.1) with an increasing F , $F(0) = 0$ is multiply subadditive on a ψ -nonatomic Boolean ring S if and only if F has the property

$$(6.2) \quad kF(\delta) < F(k\delta) \text{ for } 0 < k < 1 \text{ and all values } \delta = \psi(e), e \in S.$$

Proof. If $pe = \sum_{r=1}^p e_r$, then $e_r \subset e$, and putting $\delta_r = \psi(e_r)$, $\delta = \psi(e)$, we see that $0 < \delta_r < \delta$, $p\delta = \sum \delta_r$. If (6.2) holds and $\Phi(e)$ is defined by (6.1), we have, therefore, for $\delta > 0$,

$$p\Phi(e) = pF(\delta) = \sum \frac{\delta_r}{\delta} F(\delta) < \sum F(\delta_r) = \sum \Phi(e_r).$$

For $\delta = 0$ this inequality holds since $F(0) = 0$, so that $\Phi(e)$ is multiply subadditive.

Conversely, suppose that Φ has this property and that $\psi(e) = \delta$ for some $e \in S$; further, let $0 < k' = p/n < 1$ be a rational number and p, n be relatively prime. We decompose e into a disjoint union $e = \bigcup_{j=1}^n \bar{e}_j$ of elements \bar{e}_j with $\psi(\bar{e}_j) = \delta/n$. For any integer $1 < i < pn$ let $\bar{e}_i = \bar{e}_j$, where j is the residue of i modulo n in the interval $1 < j < n$. Then

$$e_r = \bigcup_{(r-1)p < i < rp} \bar{e}_i$$

is a disjoint union and the e_r cover e exactly p times. Moreover, $\psi(e_r) = p\delta/n = k'\delta$. Therefore,

$$pF(\delta) = p\Phi(e) < \sum \Phi(e_r) = \sum_{r=1}^p F(k'\delta) = nF(k'\delta),$$

or

$$k'F(\delta) < F(k'\delta).$$

If now k is a real number $0 < k < 1$, we take an increasing sequence of rationals $k'_n \rightarrow k$ and deduce $k'_n F(\delta) < F(k'_n \delta) < F(k\delta)$, which gives (6.2).

A function $F(u)$ satisfying (6.2) is easily seen to be continuous. Conversely, any positive, continuous, and concave function $F(u)$ satisfies (6.2). For it is known that F with $F(0) = 0$ has these properties if and only if

$$(6.3) \quad F(u) = \int_0^u f(x) dx,$$

f positive and decreasing, and this implies (6.2). There are functions of the type (6.1) which are subadditive, but not multiply subadditive. Let S be the Boolean algebra of measurable sets $e \subset (0, 1)$ and $\psi(e)$ be the Lebesgue measure of the set $e \subset (0, 1)$. Set $F(u) = \frac{2}{3}u$ in $(0, \frac{1}{3})$, $F(u) = \frac{1}{2}$ in $(\frac{1}{3}, \frac{2}{3})$, and $F(u) = \frac{2}{3}u - \frac{1}{2}$ in $(\frac{2}{3}, 1)$. Then the function (6.1) is subadditive because $F(u)$ has the property $F(u_1 + u_2) \leq F(u_1) + F(u_2)$. However, condition (6.2) is not satisfied, for $\frac{2}{3} = \frac{2}{3}F(1) > F(\frac{2}{3}) = \frac{1}{2}$.

We can also describe functions of type (6.1) by means of their representations. Assume for simplicity that $\Phi(e) = m_e$ is the Lebesgue measure of a measurable set $e \subset (0, 1)$. Let T denote one-to-one measure-preserving transformations of

$(0, 1)$ into itself, so that $e' = T(e)$ has the same measure as e for any measurable e . Then we have:

(6.4) *An increasing multiply subadditive function $\Phi(e)$ is of the form $\Phi(e) = F(me)$ if and only if Φ has a representation*

$$(6.5) \quad \Phi(e) = \sup_{\phi \in C} \phi(e),$$

where the class C contains with any $\phi(e)$ also any function $\phi(T(e))$.

If Φ has a representation of this kind, $\Phi(e)$ depends only on me , since, for any two sets e, e' with $me = me'$, there is a T with $e' = T(e)$. Therefore, $\Phi(e)$ is of the form $F(me)$. On the other hand, if a multiply subadditive and increasing function (6.5) depends only on me , we may replace C by the class C_1 of all additive functions $\phi(T(e))$, $\phi \in C$, T arbitrary, and have again

$$\Phi(e) = \sup_{\phi \in C_1} \phi(e).$$

A special case of the above class is described as follows. Let S be as before; we define the rearrangement of a set-function

$$\phi(e) = \int_e g(x) dx, \quad e \in S$$

to be any function

$$\bar{\phi}(e) = \int_e \bar{g} dx,$$

where $\bar{g}(x)$ is a rearrangement of $g(x)$ (for rearrangements of a point-function see [4, p. 276]).

(6.6) *In order that $\Phi(e)$ be of the form $\Phi(e) = \sup_C \phi(e)$, where C is the class of all rearrangements of a single, absolutely continuous positive function $\phi_0(e)$, it is necessary and sufficient that $\Phi(e) = F(me)$ where $F(u)$ is continuous, increasing and concave.*

If $\Phi(e) = \sup \phi(e)$ with the stated specification, and

$$\phi_0(e) = \int_e g dx, \quad g \geq 0,$$

then we have

$$\Phi(e) = \int_0^{me} g^*(x) dx,$$

where g^* is the decreasing rearrangement of g . Thus $\Phi(e) = F(me)$, where

$$F(u) = \int_0^u g^* dx$$

is continuous, increasing and concave. Conversely, if $\Phi(e) = F(me)$ and

$$F(u) = \int_0^u g dx$$

with an integrable, positive and decreasing g , then $\Phi(e) = \sup \phi(e)$, where $\phi(e)$ are all rearrangements of

$$\phi_0(e) = \int_0^{m_e} g dx.$$

We finally indicate a generalization of the Hahn decomposition theorem for subadditive functions. Let S be a Boolean σ -ring with zero element [2] and $\Phi(e)$ a subadditive function on S (compare §1). An element $e \in S$ is called Φ -positive, Φ -negative, or Φ -zero if $\Phi(e') > 0$, $\Phi(e') < 0$, or $\Phi(e') = 0$, respectively, for each $e' \subseteq e$, $e' \in S$. Then the following statement holds:

(6.7) If a bounded subadditive function $\Phi(e)$ on S has the property

$$(6.8) \quad \lim_{n \rightarrow \infty} \Phi(e_n) = 0, \quad e_1 \supset e_2 \supset \dots, \bigcap e_n = 0,$$

and takes values of different sign, then there are disjoint elements $e^-, e_a^+, a \in A$ of S such that e^- is Φ -negative, each e_a^+ is Φ -positive, $\Phi(e_a^+) > 0$, and each $e \in S$ disjoint with all e^- , e_a^+ is Φ -zero.

The proof is similar to the usual proof of Hahn's theorem [3, p.121], but requires transfinite induction for Φ -positive elements.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. G. Birkhoff, *Lattice theory* (2nd ed., New York, 1948).
3. P. R. Halmos, *Measure theory* (New York, 1950).
4. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge, 1934).
5. G. G. Lorentz, *On the theory of spaces Λ* , Pacific J. Math., vol. 1 (1951), 411-429.

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THE SUPREMUM OF A FAMILY OF ADDITIVE FUNCTIONS

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Summary. Any system S in which an addition is defined for some, but not necessarily all, pairs of elements can be imbedded in a natural way in a commutative semi-group G , although different elements in S need not always determine different elements in G (see § 2). Theorem 2.1 gives necessary and sufficient conditions in order that a functional $p(x)$ on S can be represented as the supremum of some family of additive functionals on S , and one such set of conditions is in terms of possible extensions of $p(x)$ to G . This generalizes the case with S a Boolean ring treated by Lorentz [4]. Lorentz imbeds the Boolean ring in a vector space and this could be done for the general S ; but we prefer to imbed S in a commutative semi-group and to give a proof (see § 1) generalizing the classical Hahn-Banach theorem to the case of an arbitrary commutative semi-group.

In § 3, S is specialized to be a relatively complemented modular lattice with zero element in which perspectivity is assumed transitive. Lemmas concerning simultaneous decompositions of several elements in S are proved which enable a certain relation in G to be described in terms of canonical decompositions in S (see Theorem 3.1). Theorem 2.1 can then be given in a more direct form for this special case generalizing the concept of "covered m times" given by Lorentz [4] for a Boolean ring.

1. The Hahn-Banach theorem for semi-groups. The theorem of Hahn-Banach concerning the extension of a linear functional [1, pp. 27-29] assumes a linear vector space. We establish now a general form of this theorem which includes the case of an arbitrary commutative group or semi-group.

T will denote an arbitrary set of real numbers t which includes the positive integers and the sum and product of any two of its elements.

A set G of elements x, y, z, \dots will be called a T -semi-group (in place of T -commutative-semi-group) if (i) $z_1 + z_2$ is defined and in G for all z_1, z_2 in G and the commutative and associative laws hold, (ii) tz is defined and in G for all z in G and t in T and the following identities hold:

$$t(z_1 + z_2) = tz_1 + tz_2, \quad (t_1 + t_2)z = t_1z + t_2z,$$

$$t_1(t_2z) = (t_1t_2)z, \quad 1z = z.$$

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In this paper function will mean one which is single-valued and has values which are finite real numbers.

A function $f(z)$ on G will be called T -additive if

$$(1.1) \quad f(z_1 + z_2) = f(z_1) + f(z_2) \quad \text{for all } z_1, z_2 \text{ in } G,$$

$$(1.2) \quad f(tz) = tf(z) \quad \text{for all } z \text{ in } G \text{ and } t \text{ in } T.$$

A function $p(z)$ on G will be called T -subadditive if

$$(1.3) \quad p(z_1 + z_2) \leq p(z_1) + p(z_2) \quad \text{for all } z_1, z_2 \text{ in } G,$$

$$(1.4) \quad p(tz) \leq tp(z) \quad \text{for all } z \text{ in } G, t \text{ in } T, t > 0.$$

In the above nomenclature the letter T may be omitted when T consists precisely of all positive integers.

Suppose now that G, G_1 are T -semi-groups with G_1 contained in G , that x_0 is in G , and that G^* consists of all y which possess a representation of at least one of the forms

$$(1.5) \quad y = x + tx_0,$$

$$(1.6) \quad y = tx_0,$$

$$(1.7) \quad y = x,$$

with x in G_1 and t in T . Suppose $h(z)$ is an arbitrary function on G and $f(x)$ a T -additive function on G_1 . A generalization to this situation of the Hahn-Banach extension lemma is given by the following theorem.

THEOREM 1.1. *Suppose that there is a function $M(u)$ on G such that*

$$(1.8) \quad f(x_2) + \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i) \geq f(x_1) + \sum_{j=1}^n (t_{1j} - t_{2j})M(u_j)$$

whenever, for arbitrary positive integers m, n ,

$$(1.9) \quad x_1 + \sum_{j=1}^n t_{1j}u_j + \sum_{i=1}^m \alpha_i z_i = x_2 + \sum_{j=1}^n t_{2j}u_j + \sum_{i=1}^m \beta_i z_i$$

with x_1, x_2 in G_1 , all u_j and z_i in G , all $t_{1j}, t_{2j}, \alpha_i, \beta_i$ in T , $t_{1j} \geq t_{2j}$ for all j , and $\alpha_i \leq \beta_i$ for all i . Then there exists a T -additive function $\phi(y)$ on G^ , which coincides with $f(x)$ on G_1 , such that (1.8) holds, with the same $M(u)$, when f, G_1 are replaced by ϕ, G^* respectively.*

Discussion of condition (1.8). The special case of (1.8) with $t_{1j} = t_{2j}$ for all j , can be stated as follows:

$$(1.10) \quad f(x_1) \leq f(x_2) + \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i)$$

whenever

$$x_1 + \sum_{i=1}^m \alpha_i z_i = x_2 + \sum_{i=1}^m \beta_i z_i$$

with $\alpha_i \leq \beta_i$ for all i .

This condition (1.10) actually implies the existence of a function $M(u)$ for which (1.8) holds, if T contains at least one negative number $-\tau$, $\tau > 0$. Indeed, from (1.9) we obtain, using arbitrary integers $p > 0$, $q_j \geq 0$,

$$\begin{aligned} p \left[x_1 + \sum_{j=1}^n t_{1j} u_j + \sum_{i=1}^m \alpha_i z_i \right] + \sum_{j=1}^n 1[-\tau u_j] + \sum_{j=1}^n q_j [-\tau u_j] \\ = p \left[x_2 + \sum_{j=1}^n t_{2j} u_j + \sum_{i=1}^m \beta_i z_i \right] + \sum_{j=1}^n (q_j + 1)[- \tau u_j], \end{aligned}$$

the term $q_j [-\tau u_j]$ to be considered absent if $q_j = 0$. Hence

$$\begin{aligned} p x_1 + \sum_{j=1}^n (p t_{1j} - q_j \tau) u_j + \sum_{j=1}^n 1[-\tau u_j] + \sum_{i=1}^m (p \alpha_i) z_i \\ = p x_2 + \sum_{j=1}^n (p t_{2j}) u_j + \sum_{j=1}^n (q_j + 1)[- \tau u_j] + \sum_{i=1}^m (p \beta_i) z_i. \end{aligned}$$

The integers p , q_j can be chosen so that for every j ,

$$2p(t_{1j} - t_{2j}) \geq q_j \tau \geq p(t_{1j} - t_{2j});$$

then (1.10) applies and yields

$$\begin{aligned} p f(x_1) &\leq p f(x_2) + \sum_{j=1}^n [q_j \tau - p(t_{1j} - t_{2j})] h(u_j) + \sum_{j=1}^n q_j h(-\tau u_j) \\ &\quad + \sum_{i=1}^m p(\beta_i - \alpha_i) h(z_i). \end{aligned}$$

Hence

$$\begin{aligned} f(x_2) + \sum_{i=1}^m (\beta_i - \alpha_i) h(z_i) &\geq f(x_1) - \sum_{j=1}^n (t_{1j} - t_{2j}) \left(\frac{q_j \tau}{p(t_{1j} - t_{2j})} - 1 \right) h(u_j) \\ &\quad - \sum_{j=1}^n (t_{1j} - t_{2j}) \frac{q_j}{p(t_{1j} - t_{2j})} h(-\tau u_j) \end{aligned}$$

so that (1.8) holds with $M(u) = - |h(u)| - (2/\tau) |h(-\tau u)|$.

Thus, in the classical Hahn-Banach lemma, where T includes all real numbers, the function $M(u)$ does not have to be mentioned explicitly in the hypotheses. In the case of a T -semi-group with T containing non-negative numbers only, condition (1.8), and the extension theorem, too, may fail even though (1.10) is valid. An example of this is given below (Example 1).

We note that (1.10), and hence (1.8) too, include the restriction

$$(1.11) \quad f(x_1) = f(x_2)$$

whenever $x_1 + z = x_2 + z$ with z in G . Also, the choice $x_1 = x + x$, $x_2 = x$, $m = 1$, $z_1 = x$, $\alpha_1 = 1$, $\beta_1 = 2$ shows that (1.10) includes the condition

$$(1.12) \quad f(x) \leq h(x) \quad \text{for all } x \text{ in } G_1.$$

If T contains $t_1 - t_2$ whenever it contains t_1 , t_2 with $t_1 > t_2$, the condition (1.8) simplifies to

$$(1.13) \quad f(x_2) + \sum_{i=1}^m \gamma_i h(z_i) \geq f(x_1) + \sum_{j=1}^n t_j M(u_j)$$

whenever, for arbitrary non-negative integers m, n ,

$$(1.14) \quad x_1 + \sum_{j=1}^n t_j u_j + v = x_2 + \sum_{i=1}^m \gamma_i z_i + v$$

with x_1, x_2 in G_1 , all t_j, γ_i in T and $> 0, v$ and all u_j, z_i in G (the terms

$$\sum_{i=1}^m \gamma_i h(z_i), \quad \sum_{j=1}^n t_j M(u_j)$$

to be replaced by 0 when m, n , respectively, take the value 0). For such T , if $h(z)$ happens to be T -subadditive, it is sufficient that there be a function $M(u)$ with the properties

$$(1.15) \quad M(tu) \geq tM(u) \quad \text{for all } u \text{ in } G, t \text{ in } T, t > 0,$$

$$(1.16) \quad M(u_1 + u_2) \geq M(u_1) + M(u_2) \quad \text{for all } u_1, u_2 \text{ in } G,$$

such that

$$(1.17) \quad f(x_2) + h(z) \geq f(x_1) + M(u)$$

whenever $x_1 + u + v = x_2 + z + v$ with x_1, x_2 in G_1 and u, z, v in G (the terms $h(z), M(u)$ to be replaced by 0 if z, u respectively are absent in the equality). Finally, for such T , if T contains at least one negative number and $h(z)$ happens to be T -subadditive, it is sufficient, without postulating the function $M(u)$, that

$$(1.18) \quad f(x_1) \leq f(x_2) + h(z)$$

whenever $x_1 + v = x_2 + z + v$ with x_1, x_2 in G_1 and z, v in G ($h(z)$ to be replaced by 0 if z is absent in the equality).

Proof of Theorem 1.1. Consider separately two cases.

Case 1. For some λ_1, λ_2 in T with $\lambda_1 \neq \lambda_2$ and for some g_1, g_2 in G_1 and v in G ,

$$(1.19) \quad \lambda_1 x_0 + g_1 + v = \lambda_2 x_0 + g_2 + v = w,$$

say. We may suppose $\lambda_1 > \lambda_2$. Set $r_0 = [f(g_2) - f(g_1)] / (\lambda_1 - \lambda_2)$ and define

$$(1.20) \quad \begin{aligned} \phi(y) &= f(x) + tr_0 & \text{if } y \text{ is given by (1.5),} \\ \phi(y) &= tr_0 & \text{if } y \text{ is given by (1.6),} \\ \phi(y) &= f(x) & \text{if } y \text{ is given by (1.7).} \end{aligned}$$

That this ϕ is single-valued and satisfies (1.8) on G^* can be seen as follows: suppose, corresponding to (1.9),

$$(1.21) \quad y_1 + \sum_{j=1}^n t_1 u_j + \sum_{i=1}^m \alpha_i z_i = y_2 + \sum_{j=1}^n t_2 u_j + \sum_{i=1}^m \beta_i z_i$$

with y_1, y_2 in G^* . If $y_1 = x_1 + t_1 x_0$ and $y_2 = x_2 + t_2 x_0$ we multiply (1.21) by λ_1 and by λ_2 and combine to obtain

$$\begin{aligned} \lambda_1 \left(x_1 + t_1 x_0 + \sum_{j=1}^n t_1 u_j + \sum_{i=1}^m \alpha_i z_i \right) + \lambda_2 \left(x_2 + t_2 x_0 + \sum_{j=1}^n t_2 u_j + \sum_{i=1}^m \beta_i z_i \right) \\ + (t_1 + t_2)(g_1 + g_2 + v) \end{aligned}$$

$$= \lambda_1 \left(x_2 + t_2 x_0 + \sum_{j=1}^n t_{2j} u_j + \sum_{i=1}^m \beta_i z_i \right) + \lambda_2 \left(x_1 + t_1 x_0 + \sum_{j=1}^n t_{1j} u_j + \sum_{i=1}^m \alpha_i z_i \right) \\ + (t_1 + t_2)(g_1 + g_2 + v),$$

that is,

$$\lambda_1 x_1 + \lambda_2 x_2 + t_1 g_1 + t_2 g_2 + \sum_{j=1}^n (\lambda_1 t_{1j} + \lambda_2 t_{2j}) u_j + \sum_{i=1}^m (\lambda_1 \alpha_i + \lambda_2 \beta_i) z_i + (t_1 + t_2) w \\ = \lambda_1 x_2 + \lambda_2 x_1 + t_1 g_1 + t_2 g_2 + \sum_{j=1}^n (\lambda_1 t_{2j} + \lambda_2 t_{1j}) u_j \\ + \sum_{i=1}^m (\lambda_1 \beta_i + \lambda_2 \alpha_i) z_i + (t_1 + t_2) w.$$

Now (1.8) for f on G_1 applies and gives

$$f(\lambda_1 x_2 + \lambda_2 x_1 + t_1 g_1 + t_2 g_2) + \sum_{i=1}^m (\lambda_1 - \lambda_2)(\beta_i - \alpha_i)h(z_i) \\ \leq f(\lambda_1 x_1 + \lambda_2 x_2 + t_1 g_2 + t_2 g_1) + \sum_{j=1}^n (\lambda_1 - \lambda_2)(t_{1j} - t_{2j})M(u_j).$$

From this follows at once

$$(1.22) \quad \phi(y_2) + \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i) \geq \phi(y_1) + \sum_{j=1}^n (t_{1j} - t_{2j})M(u_j).$$

Similar reasoning shows that (1.21) implies (1.22) if y_1, y_2 have representations of any of the forms (1.5), (1.6), (1.7). This implies that ϕ is single-valued and satisfies (1.8) on G^* . It is evident that ϕ is T -additive and coincides with f on G_1 , so that Theorem 1.1 is proved for Case 1.

Case 2. In every relation of the form (1.19), $\lambda_1 = \lambda_2$. Then, with a number r_0 to be assigned later, we define $\phi(y)$ as in (1.20). Irrespective of the value of r_0 , this ϕ is single-valued on G^* . For suppose $y_1 = y_2$. If $y_1 = x_1 + t_1 x_0$ and $y_2 = x_2 + t_2 x_0$ then $t_1 x_0 + x_1 + v = t_2 x_0 + x_2 + v$ for any v in G , hence (this is Case 2) $t_1 = t_2$ and, using (1.11), $f(x_1) = f(x_2)$, $\phi(y_1) = \phi(y_2)$. Similar reasoning applies if y_1, y_2 have representations of any of the forms (1.5), (1.6), (1.7) to show that ϕ is single-valued on G^* . It is evident that ϕ is T -additive and coincides with f on G_1 .

Thus we need only show that an r_0 exists for which (1.21) implies (1.22) with arbitrary y_1, y_2 in G^* . It is easily seen that it is sufficient to do this for the y_1, y_2 with representations $y_1 = x_1 + t_1 x_0$, $y_2 = x_2 + t_2 x_0$ with $t_1 \neq t_2$. There are therefore two conditions to satisfy, according as $t_1 > t_2$ or $t_2 > t_1$. Explicitly, we require (use a bar to distinguish the two possibilities),

$$(1.23) \quad \frac{-1}{(\bar{t}_2 - \bar{t}_1)} \left[f(\bar{x}_2) - f(\bar{x}_1) + \sum_{i=1}^m (\bar{\beta}_i - \bar{\alpha}_i)h(\bar{z}_i) - \sum_{j=1}^n (\bar{t}_{1j} - \bar{t}_{2j})M(\bar{u}_j) \right] < r_0$$

whenever

$$(1.24) \quad \begin{cases} \bar{l}_2 > \bar{l}_1 \\ \bar{x}_1 + \bar{l}_1 x_0 + \sum_{j=1}^{\bar{n}} \bar{l}_{1j} \bar{u}_j + \sum_{i=1}^{\bar{m}} \bar{a}_i \bar{x}_i = \bar{x}_2 + \bar{l}_2 x_0 + \sum_{j=1}^{\bar{n}} \bar{l}_{2j} \bar{u}_j + \sum_{i=1}^{\bar{m}} \bar{\beta}_i \bar{x}_i, \end{cases}$$

and

$$(1.25) \quad r_0 < \frac{1}{(l_1 - l_2)} \left[f(x_2) - f(x_1) + \sum_{i=1}^m (\beta_i - \alpha_i) h(z_i) - \sum_{j=1}^n (l_{1j} - l_{2j}) M(u_j) \right]$$

whenever

$$(1.26) \quad \begin{cases} l_1 > l_2 \\ x_1 + l_1 x_0 + \sum_{j=1}^n l_{1j} u_j + \sum_{i=1}^m \alpha_i x_i = x_2 + l_2 x_0 + \sum_{j=1}^n l_{2j} u_j + \sum_{i=1}^m \beta_i x_i. \end{cases}$$

That {L.H.S. of (1.23)} < {R.H.S. of (1.25)} follows from (1.24) and (1.26), using (1.8) for f on G_1 . Hence

$$\sup \{ \text{L.H.S. of (1.23)} \} < \inf \{ \text{R.H.S. of (1.25)} \}$$

showing that r_0 exists, as required, if there are realizations of (1.24) and (1.26). Now there are realizations of (1.26), for example: $x_1 = x_2$ (an arbitrary element in G_1),

$$l_1 = 2, \quad l_2 = 1, \quad n = m = 1, \quad u_1 = z_1 = x_0, \quad l_{11} = l_{21} = 1, \quad \alpha_1 = 1, \quad \beta_1 = 2.$$

There are also realizations of (1.24) (it was to ensure this that the function $M(u)$ was postulated¹), for example: $x_1 = x_2$ (an arbitrary element in G_1),

$$\bar{l}_1 = 1, \quad \bar{l}_2 = 2, \quad \bar{n} = \bar{m} = 1, \quad \bar{u}_1 = \bar{z}_1 = x_0, \quad \bar{l}_{11} = 2, \quad \bar{l}_{21} = 1, \quad \bar{\alpha}_1 = \bar{\beta}_1 = 1.$$

This proves Theorem 1.1 for Case 2 and completes the proof of the theorem.

COROLLARY. Under the conditions of Theorem 1.1 the T -additive function $f(x)$ can be extended by transfinite induction to a T -additive function $\phi(z)$ on G such that (use (1.8) for ϕ on G) $M(z) < \phi(z) < h(z)$ for all z in G .

THEOREM 1.2. Let $h(z)$ be a function on a T -semi-group G such that, for some function $M(u)$,

$$(1.27) \quad (l_2 - l_1)h(z) + \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i) > \sum_{j=1}^n (l_{1j} - l_{2j})M(u_j)$$

whenever, for arbitrary positive integers m, n ,

$$(1.28) \quad l_1 z + \sum_{j=1}^n l_{1j} u_j + \sum_{i=1}^m \alpha_i z_i = l_2 z + \sum_{j=1}^n l_{2j} u_j + \sum_{i=1}^m \beta_i z_i,$$

with z , all u_j , all z_i in G , l_1, l_2 , all l_{1j}, l_{2j} , α_i, β_i in T , $l_{1j} > l_{2j}$, $\alpha_i < \beta_i$. Then for arbitrary (but fixed) x_0 in G there is a T -additive function $\phi(z)$ on G with $\phi(x_0) = h(x_0)$ and $M(z) < \phi(z) < h(z)$ for all z in G .

¹In the classical Hahn-Banach theorem for linear vector spaces, $h(z)$ is a subadditive function $p(z)$ with $p(tz) = tp(z)$ for all $t > 0$ and $-p(-u)$ acts as the function $M(u)$ which we postulated explicitly. G. G. Lorentz has independently had the idea of investigating extensions of an additive $f(x)$ satisfying $q(x) < f(x) < p(x)$ for given subadditive $p(z)$ and superadditive $q(z)$.

Remark. The hypotheses imply:

(1.29) $h(z)$ is T -subadditive and $h(tz) = th(z)$ for all z in G , t in T , $t > 0$,

(1.30) $h(z_1) = h(z_2)$ whenever $z_1 + v = z_2 + v$ with z_1, z_2, v in G .

Proof. Let G_1 be the T -semi-group of all tx_0 with t in T and define $f(x)$, T -additive on G_1 , by $f(tx_0) = th(x_0)$. This f is single-valued, for if $t_1x_0 = t_2x_0$ the hypotheses of the theorem imply that $t_1h(x_0) = t_2h(x_0)$. It is also evident from (1.27) that (1.9) implies (1.8) in the present situation. Thus Theorem 1.1 applies to extend f to a ϕ with the required properties.

THEOREM 1.3. *The hypotheses of Theorem 1.2 are necessary and sufficient in order that $h(z)$ admit a representation*

$$(1.31) \quad h(z) = \sup\{\phi_\lambda(z)\}$$

with a family of T -additive functions ϕ_λ for which $\inf\{\phi_\lambda(u)\}$ is finite for every u in G .

Proof. The hypotheses of Theorem 1.2 imply a representation (1.31), in fact with $h(z) = \max\{\phi(z)\}$ for a family of T -additive $\phi(z)$ with $M(u) \leq \phi(u)$ for all ϕ in the family and all u in G .

Conversely, if there is a representation (1.31), then for each λ ,

$$t_1\phi_\lambda(z) + \sum_{j=1}^n t_{1j}\phi_\lambda(u_j) + \sum_{i=1}^m \alpha_i\phi_\lambda(z_i) = t_2\phi_\lambda(z) + \sum_{j=1}^n t_{2j}\phi_\lambda(u_j) + \sum_{i=1}^m \beta_i\phi_\lambda(z_i).$$

Hence (1.27) holds with $M(u) = \inf\{\phi_\lambda(u)\}$.

COROLLARY 1. *If $h(z)$ admits a representation (1.31) it admits such a representation with sup replaced by max (possibly with a different family of T -additive functions ϕ_λ).*

COROLLARY 2. *The $M(u)$ in (1.27) may be restricted to functions satisfying (1.15), (1.16).*

THEOREM 1.4. *If T contains $t_1 - t_2$ whenever it contains t_1, t_2 with $t_1 > t_2$, then necessary and sufficient conditions that $h(z)$ admit a representation (1.31) are: (1.29) and*

(1.32) *for some $M(u)$ satisfying (1.15), (1.16), $h(z_1) \geq h(z_2) + M(u)$ whenever $z_1 + v = z_2 + u + v$ (with $h(z_1), h(z_2), M(u)$ replaced by 0 if z_1, z_2, u respectively are absent in the equality).*

Proof. This follows easily from Theorem (1.3).

Remark. For the particular case when T consists precisely of all positive integers, (1.29) can be replaced by

$$(1.33) \quad h(z_1 + z_2) \leq h(z_1) + h(z_2) \quad \text{for all } z_1, z_2 \text{ in } G,$$

$$(1.34) \quad h(z + z) = h(z) + h(z) \quad \text{for all } z \text{ in } G.$$

To prove this we need only show that (1.33) implies $h(nz) = nh(z)$ for all positive integers n . But repeated applications of (1.33) give $h(nz) \leq nh(z)$ and repeated applications of (1.34) give $h(2^m z) = 2^m h(z)$. By choosing $2^m > n$ we obtain

$$h(2^m z) \leq h((2^m - n)z) + h(nz)$$

by (1.33), and hence

$$2^m h(z) \leq (2^m - n)h(z) + h(nz),$$

from which follows $nh(z) \leq h(nz)$ and therefore $h(nz) = nh(z)$, as required.

THEOREM 1.5. *If T contains at least one negative number $-\tau$, $\tau > 0$, then necessary and sufficient conditions that $h(z)$ admit a representation (1.31), whether $\inf \{\phi_\lambda(u)\}$ is required to be finite or not, are the same, namely:*

$$(t_1 - t_2)h(z) \leq \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i)$$

whenever, for a positive integer m ,

$$t_1 z + \sum_{i=1}^m \alpha_i z_i = t_2 z + \sum_{i=1}^m \beta_i z_i,$$

with z , all z_i in G , t_1, t_2 all α_i, β_i in T , $\alpha_i \leq \beta_i$.

Proof. The methods used on page 465 in the discussion of condition (1.8), Theorem 1.1, show that, with the present hypotheses, (1.28) implies (1.27) if $M(u)$ is taken as $-|h(u)| - (2/\tau)|h(-\tau u)|$.

COROLLARY. *If T contains $t_1 - t_2$ whenever it contains t_1, t_2 with $t_1 > t_2$ and T also contains at least one negative number, then necessary and sufficient conditions that $h(z)$ admit a representation (1.31) are:*

$$\begin{aligned} h(z_1 + z_2) &\leq h(z_1) + h(z_2) && \text{for all } z_1, z_2 \text{ in } G, \\ h(tz) &= th(z) && \text{for all } z \text{ in } G, t \text{ in } T, t > 0, \\ h(z_1) &= h(z_2) && \text{whenever } z_1 + v = z_2 + v, z_1, z_2, v \text{ in } G. \end{aligned}$$

The following examples show the necessity of postulating the function $M(u)$ in Theorem 1.1, and the finiteness of $\inf \{\phi_\lambda(u)\}$ in the representation (1.31).

Example 1. T consists of all real non-negative numbers; G consists of all two-dimensional vectors $[a_1, a_2]$ with $a_1 \geq 0, a_2 \geq 0$; G_1 consists of all $[a_1, 0]$ and $x_0 = [0, 1]$; $h[a_1, a_2] = a_2$ if $a_2 > 0$ and $h[a_1, a_2] = a_1$ if $a_2 = 0$; $f[a_1, 0] = a_1$.

Then f is T -additive on G_1 and condition (1.10) is satisfied. But for any T -additive extension ϕ of f ($G^* = G$ in this example) and for every positive integer n , $\phi[n, 1] = n + \phi(x_0)$, whereas $h[n, 1] = 1$ so that there is no such ϕ with $\phi[n, 1] \leq f[n, 1]$ for all n . Thus Theorem 1.1 cannot be proved on the basis of (1.10) alone.

In this example h is T -subadditive and satisfies (1.29), (1.30), yet h does not admit a representation (1.31) (even with $\inf \{\phi_\lambda(u)\}$ unrestricted). For if ϕ

is T -additive and $\phi[a_1, a_2] \leq h[a_1, a_2]$ then

$$\phi[n, 1] = n\phi[1, 0] + \phi[0, 1] \leq 1 \quad \text{for all } n;$$

hence $\phi[1, 0] \leq 0$ for all such ϕ , whereas $h[1, 0] = 1$.

Example 2. T consists of all non-negative integers; G consists of all infinite-dimensional vectors $a = (a_0, a_1, \dots, a_m, \dots)$ with every a_m a non-negative integer and at most a finite number of a_m different from 0; $h(a) = \max \{m(a_m - a_0)\}$.

Then $h(a) = \sup\{\phi_\lambda(a)\}$ with $\phi_\lambda(a)$ the T -additive function $\lambda(a_\lambda - a_0)$ ($\lambda = 0, 1, 2, \dots$). Nevertheless h does not admit a representation (1.31) with $\inf\{\phi_\lambda(a)\}$ finite. To see this, let a^n denote the vector with $(a^n)_m = 0$ for $m \neq n$ and $(a^n)_m = 1$ for $m = n$. If $\phi(a)$ is a T -additive function with $\phi(a) \leq h(a)$ for all a then

$$\phi(a^n) + \phi(a^0) = \phi(a^n + a^0) \leq h(a^n + a^0) = 0.$$

Hence if $h(a^n) = \sup\{\phi(a^n)\}$ for every n it would follow that $\inf\{\phi(a^0)\} \leq -n$ for every n .

An elegant generalization (in a different way) of the classical Hahn-Banach theorem has been given by Hidegoro Nakano [5, pp. 89-91]. Nakano deals with a linear vector space, that is, with all real numbers as scalar multipliers, but for given $h(z)$ and x_0 , the requirements that there shall be a T -additive ϕ with $\phi(x_0) = h(x_0)$, $\phi(z) \leq h(z)$ for all z , are replaced by the requirements that there shall be a T -additive ϕ with $\phi(z) \leq \phi(x_0) - h(x_0) + h(z)$ for all z .

Theorems 1.1 to 1.5 of the present paper can be extended to include Nakano's generalization.

THEOREM 1.6. *In order that $h(z)$ admit a representation*

$$(1.35) \quad h(z) = \sup\{A_\lambda + \phi_\lambda(z)\}$$

with a family of T -additive ϕ_λ and constants A_λ for which $|A_\lambda| \leq K < \infty$ for all λ and $\inf\{\phi_\lambda(u)\}$ is finite for every u in G , it is necessary and sufficient that functions $A(u)$, $M(u)$ exist with $|A(u)| \leq K$ for all u and

$$(1.36) \quad (t_2 - t_1)h(z) + \sum_{i=1}^m (\beta_i - \alpha_i)h(z_i) \\ > \sum_{j=1}^n (t_{1j} - t_{2j})M(u_j) + \left(t_2 - t_1 + \sum_{i=1}^m (\beta_i - \alpha_i) - \sum_{j=1}^n (t_{1j} - t_{2j}) \right) A(z)$$

whenever (1.28) holds.

Proof. If (1.28) implies (1.36), the argument used in the proof of Theorem 1.2 shows that for every x_0 in G there is a T -additive ϕ_0 such that $\phi_0(x_0) = h(x_0) - A(x_0)$ and

$$M(z) - A(x_0) \leq \phi_0(z) \leq h(z) - A(x_0)$$

for all z in G . Hence (1.35) holds with these functions $A(x_0) + \phi_0(z)$.

Conversely if (1.35) does hold then (1.28) implies (1.36) if $M(u)$ is taken to be $\inf\{A_\lambda + \phi_\lambda(u)\}$ and $A(z)$ is taken to be the limit of A_{λ_n} for any sequence of λ_n for which $A_{\lambda_n} + \phi_{\lambda_n}(z)$ converges to $h(z)$ and A_{λ_n} converges, as n becomes infinite.

Remark. If $h(z)$ admits a representation (1.35) then h is T -convex, that is,

$$(1.37) \quad h(ax + (1-a)y) \leq ah(x) + (1-a)h(y)$$

whenever x, y are in G and $a, 1-a$ are in T ($0 \leq a \leq 1$).

2. Systems S with partially-defined addition operator. Now let S be any system of elements, a, b, c, \dots with an addition $a \dot{+} b$ defined, and in S , for some, but not necessarily all, ordered pairs a, b in S . No further properties of $\dot{+}$ will be postulated in this section. We shall call a function $\phi(a)$ on S additive if $\phi(a \dot{+} b) = \phi(a) + \phi(b)$ whenever $a \dot{+} b$ is defined.

Let G be the set of all formal sums $x = a_1 + \dots + a_r$, with an arbitrary (but finite) number of a_i from S , the order being immaterial by definition and with two sums x, y identified in G ($x = y$) if x can be transformed into y by a finite number of changes of the form: a is replaced by $a_1 + a_2$ or conversely $a_1 + a_2$ is replaced by a if $a_1 \dot{+} a_2 = a$. If $x = a_1 + \dots + a_r$, and $y = b_1 + \dots + b_n$, let the definition of $x + y$ in G be

$$x + y = a_1 + \dots + a_r + b_1 + \dots + b_n.$$

Then G is a semi-group and each element a in S determines an element $x = a$ in G . We shall say S determines G .

THEOREM 2.1. *A function $p(a)$ on S admits a representation*

$$(2.1) \quad p(a) = \sup\{\phi_\lambda(a)\}$$

(ϕ additive on S , $\inf\{\phi_\lambda(a)\}$ finite for each a in S) if and only if it admits a representation

$$(2.2) \quad p(a) = \max\{\psi_\lambda(a)\}$$

(ψ_λ additive on S , $\inf\{\psi_\lambda(a)\}$ finite for each a in S) and if and only if $p(a)$ has an extension $p_1(x)$ defined for all x in the G determined by S so that p_1 satisfies (1.32), (1.33), and (1.34), and if and only if $p(a)$ has the two properties:

$$(2.3) \quad mp(a) \leq p(a_1) + \dots + p(a_r)$$

whenever $ma + u = a_1 + \dots + a_r + u$ in G ;

$$(2.4) \quad \inf\{m^{-1}(p(a_1) + \dots + p(a_r) - p(b_1) - \dots - p(b_n))\} > -\infty$$

whenever, for fixed c_1, \dots, c_n , the integers m, r, n and the $a_1, \dots, a_r, b_1, \dots, b_n$ vary so that

$$m(c_1 + \dots + c_n) + b_1 + \dots + b_n = a_1 + \dots + a_r.$$

(In connection with this theorem see Lorentz [4]. The definition of $p_1(x)$ in (2.5) below was suggested by [4].)

Proof. For each additive $\phi(a)$ on S define $\phi_1(x) = \phi(a_1) + \dots + \phi(a_r)$ if $x = a_1 + \dots + a_r$. Then ϕ_1 is single-valued and additive on G and is an extension of ϕ . Hence if $p(a)$ does admit a representation (2.1) then the function

$$p_1(x) = \sup\{\phi_{\lambda 1}(x)\}$$

is an extension of $p(a)$ which, by Theorem 1.4, satisfies (1.32), (1.33), (1.34). On the other hand, if $p(a)$ has any extension $p_1(x)$ which satisfies these conditions, then by Theorem 1.4, $p_1(x)$ admits a representation (1.31) on G , which, when considered on S only, gives a representation (2.2) for $p(a)$ on S .

Again if $p_1(x)$ on G satisfies (1.32), (1.33), (1.34), then clearly it satisfies (2.3) and (2.4). If such a $p_1(x)$ is an extension of $p(a)$ then $p(a)$ must satisfy (2.3) and (2.4). Conversely, if $p(a)$ on S satisfies (2.3) and (2.4) we define

$$(2.5) \quad p_1(x) = \inf\{m^{-1}(p(a_1) + \dots + p(a_r))\}$$

for all a_1, \dots, a_r with $mx + u = a_1 + \dots + a_r + u$ for some positive integer m and some u in G . Then (2.3) ensures that $p_1(x)$ is an extension of $p(a)$, (2.4) ensures that $p_1(x)$ has finite real numbers as values, and from (2.5) it follows that (1.32), (1.33), (1.34), with $M(c_1 + \dots + c_s) = \text{L.H.S. of (2.4)}$, hold for $p_1(x)$.

Remark. If the cancellation law, $x + u = y + u$ implies that $x = y$, holds in G , the condition (2.3) is equivalent to the (apparently) weaker condition

$$(2.6) \quad mp(a) \leq p(a_1) + \dots + p(a_r)$$

whenever $ma = a_1 + \dots + a_r$ in G (see the definition of multiple subadditivity given in [4]).

3. Modular lattices with zero and relative complements. Suppose now that S is a modular (but not necessarily distributive) relatively complemented lattice with zero element 0, so that the von Neumann theory of "independence" (or "independence over 0" in terms of [3]) is valid at least for finite collections of elements of S [6; 7; 3, p. 539; 2, p. 114]. Suppose too that $e_1 \dot{+} e_2$ is identical with lattice union $e_1 \cup e_2$ restricted to independent elements.

We recall that

$$e = \bigcup_{i=1}^n e_i$$

is called a direct decomposition if e_1, \dots, e_n are independent and e is called perspective to f (with axis a) written $e \smile f$, if $e \cup a = f \cup a$ and $e \cap a = f \cap a = 0$ for some a in S .

In what follows we shall postulate that S has the additional property that perspectivity is transitive, that is,

$$(3.1) \quad e \smile f, \quad f \smile g \quad \text{imply} \quad e \smile g.$$

(In a Boolean ring (3.1) holds trivially since $e \smile f$ implies $e = f$. But (3.1) holds also for the continuous geometries of von Neumann or more generally [6; 7; 3])

if S has certain continuity properties.) With the hypothesis (3.1) we shall show, for given $e_1, \dots, e_n, f_1, \dots, f_m$ in S , that equality in G ,

$$e_1 + \dots + e_n + h_1 + \dots + h_p = f_1 + \dots + f_m + h_1 + \dots + h_p$$

for some h_1, \dots, h_p in S , can be expressed in a simple way in terms of direct decompositions of $e_1, \dots, e_n, f_1, \dots, f_m$.

LEMMA 1. Suppose that

$$e = \bigcup_{i=1}^n e_i$$

is a direct decomposition and that $e \sim f$. Then there exists a direct decomposition $f = \bigcup f_i$ with $e_i \sim f_i$ for each i .

LEMMA 2. Suppose that $e = e_1 \cup e_2$, $f = f_1 \cup f_2$ are direct decompositions with $e \sim f$ and $e_1 \sim f_1$. Then $e_2 \sim f_2$.

LEMMA 3 (Additivity of perspectivity). Suppose that

$$e = \bigcup_{i=1}^n e_i \quad \text{and} \quad f = \bigcup_{i=1}^n f_i$$

are direct decompositions with $e_i \sim f_i$ for each i . Then $e \sim f$.

Under stronger assumptions these lemmas were proved in [7] but the proofs are valid without change in the present case. Lemma 1 corresponds to a corollary of [3, Lemma 3.3] and Lemmas 2 and 3 correspond to [3, Lemmas 6.2, 6.4].

LEMMA 4. Suppose f_1, \dots, f_m and e are arbitrary. Then there exist direct decompositions $f_j = f_{1j} \cup f'_{1j}$, $e = e_1 \cup \dots \cup e_{m+1}$ such that $e_j \sim f_{1j}$ for $1 \leq j \leq m$ and $e_1 \cup \dots \cup e_m = e \cap (f_1 \cup \dots \cup f_m)$.

Proof. The lemma can be verified as follows: Let $a_j = f_1 \cup \dots \cup f_{j-1}$ for $1 \leq j \leq m+1$ and let $a_1 = 0$. Replacing f_j for $1 \leq j \leq m$ by a complement of $a_j \cap f_j$ with respect to f_j we may, and shall, suppose that f_1, \dots, f_m are independent. Set $e_1 = e \cap f_1$; for $1 \leq j \leq m$ set e_j equal to a complement of $e \cap a_j$ with respect to $e \cap a_{j+1}$; set e_{m+1} equal to a complement of $e \cap a_{m+1}$ with respect to e ; set $f_{11} = e_1$; for $1 \leq j \leq m$ set $f_{1j} = f_j \cap (e_j \cup a_j)$; for $1 \leq j \leq m$ set f'_{1j} equal to a complement of f_{1j} with respect to f_j .

We shall show that $e_j \sim f_{1j}$ with axis a_j . This is trivial for $j = 1$ and for $j > 1$ we have

$$\begin{aligned} f_{1j} \cup a_j &= (f_j \cap (e_j \cup a_j)) \cup a_j \\ &= (f_j \cup a_j) \cap (e_j \cup a_j) = e_j \cup a_j \end{aligned}$$

by the modular law and since $e_j \leq f_j \cup a_j = a_{j+1}$ and $a_j \leq f_j \cup a_j$. On the other hand,

$$f_{1j} \cap a_j = f_{1j} \cap f_j \cap a_j = 0$$

since the f_1, \dots, f_j are independent and $e_j \cap a_j = e_j \cap (e \cap a_j) = 0$. This proves that $e_j \sim f_{1j}$. The other parts of the lemma are easily verified.

LEMMA 5. Suppose e_1, \dots, e_n are arbitrary. Then there are independent elements g_j ($j = 1, \dots, N_n$) and direct decompositions

$$e_1 = g_1 \cup \dots \cup g_{N_1},$$

$$e_2 = g_1^{(2)} \cup \dots \cup g_{N_1}^{(2)} \cup g_{N_1+1} \cup \dots \cup g_{N_2},$$

$$e_n = g_1^{(n)} \cup \dots \cup g_{N_{n-1}}^{(n)} \cup g_{N_{n-1}+1} \cup \dots \cup g_{N_n},$$

such that

$$\bigcup_{j=1}^{N_{r-1}} g_j^{(r)} = e_r \cap (e_1 \cup \dots \cup e_{r-1}),$$

$$g_j^{(r)} = 0 \quad \text{or} \quad g_j^{(r)} \sim g_j,$$

for $1 < r \leq n$ and $1 \leq j \leq N_{r-1}$.

Proof. This lemma can be verified by induction on n , using Lemma 4.

LEMMA 6 (Superposition of decompositions). Suppose that

$$e = \bigcup_{i=1}^n e_i \quad \text{and} \quad f = \bigcup_{j=1}^m f_j$$

are direct decompositions and that $e \sim f$. Then there exist direct decompositions $e_1 = \bigcup e_{1j}$, $f_j = \bigcup f_{1j}$ such that $e_{1j} \sim f_{1j}$ for all i, j .

Proof. We shall assume, as we clearly may by Lemma 1 and the transitivity of perspectivity, that $e = f$. Apply Lemma 4 to f_1, \dots, f_m and e_1 (in place of e) and obtain the direct decompositions

$$e_1 = \bigcup_{j=1}^m e_{1j}, \quad f_j = f_{1j} \cup f_j' \quad \text{with} \quad e_{1j} \sim f_{1j}.$$

By Lemma 3, $e_1 \sim \bigcup f_{1j}$ and hence by Lemma 2 $(e_2 \cup \dots \cup e_n) \sim \bigcup f_j'$. This means that the lemma for n has been reduced to the lemma for $n-1$. By successive reductions the lemma can be reduced to the case $n=1$ and for this case the lemma holds by Lemma 1.

THEOREM 3.1. If $x = e_1 + \dots + e_n$ and $y = f_1 + \dots + f_m$, then $x + u = y + u$ for some u in G if and only if there exist independent elements g_1, \dots, g_N and direct decompositions

$$(3.2) \quad e_i = \bigcup_{j=1}^N e_{ij}, \quad f_i = \bigcup_{j=1}^N f_{ij},$$

such that each e_{ij} is either 0 or $\sim g_j$, each f_{ij} is either 0 or $\sim g_j$, and for each j the number E_j of i for which $e_{ij} \sim g_j$ is equal to the number F_j of i for which $f_{ij} \sim g_j$.

Proof. Write $x \sim y(d)$ if decompositions (3.2) do exist and write $x = y(c)$ if $x + u = y + u$ for some u in G . Since $e \sim f$ implies $e + a = f + a$ for some axis of perspectivity a in S , it follows that $e \sim f$ implies that $e = f(c)$ and hence $x \sim y(d)$ implies $x = y(c)$.

The converse, $x \equiv y(c)$ implies $x \smile y(d)$, will follow by induction if we prove:

$$(3.3) \quad x \smile x(d);$$

(3.4) if $x \smile y(d)$, this relation remains valid if f_1 in y is replaced by $f' + f''$, providing that $f_1 = f' + f''$;

(3.5) if $x \smile y(d)$, this relation remains valid if $f_1 + f_2$ in y is replaced by f , providing that $f_1 + f_2 = f$;

$$(3.6) \quad \text{if } x + u \smile y + u(d) \text{ then } x \smile y(d).$$

For $x + u \equiv y + u$ means that $x + u$ can be transformed into $y + u$ by the changes named in (3.3), (3.4), and (3.5) and it will follow that $x + u \smile y + u(d)$. From (3.6) we will then have $x \smile y(d)$ as required.

Proof of (3.3). Given arbitrary elements e_1, \dots, e_n we need only show that there are independent elements g_1, \dots, g_N and direct decompositions

$$e_i = \bigcup_{j=1}^N e_{ij}$$

such that each e_{ij} is either 0 or $\smile g_j$. But this follows from Lemma 5.

Proof of (3.4). Suppose $x \smile y(d)$. This implies an independent set g_1, \dots, g_N and a particular decomposition (we shall call it the previous decomposition) for each f_i in y . If now f_1 is replaced by $f' + f''$, then Lemma 6 can be applied to the previous decomposition of f_1 , say $f_1 = \bigcup f_{1j}$, and the decomposition $f' \cup f''$ of f_1 . Direct decompositions $f_{1j} = f_{1j}' \cup f_{1j}''$ result, and these, with the help of Lemma 1, lead to direct decompositions $g_j = g_j' \cup g_j''$ with $f_{1j}' \smile g_j'$, $f_{1j}'' \smile g_j''$ if f_{1j} is different from 0 and with $g_j' = g_j$, $g_j'' = 0$ if $f_{1j} = 0$. From these decompositions of g_j we obtain direct decompositions, $f_{ij} = f_{ij}' \cup f_{ij}''$ for $i > 1$ and $e_{ij} = e_{ij}' \cup e_{ij}''$ so that $x \smile y(d)$ remains valid with g_1', \dots, g_N' , g_1'', \dots, g_N'' in place of g_1, \dots, g_N .

Proof of (3.5). Suppose $x \smile y(d)$, that the $e_{ij}, f_{ij}, e_{ij}, g_j$ satisfy (3.2), and that $f_1 + f_2$ in y is replaced by f . (Note that

$$f = \left(\bigcup_{j=1}^N f_{1j} \cup \bigcup_{j=1}^N f_{2j} \right)$$

is a direct decomposition for f ; but this fails to prove that $x \smile y(d)$ remains valid with the same g_1, \dots, g_N since, for some j , both f_{1j} and f_{2j} may differ from zero.) We may suppose that all g_j are different from 0, that $f_{1j} = g_j$ for $j = 1, \dots, p$ (in place of $f_{1j} \smile g_j$), and that $f_{1j} = 0$ for $j > p$ (apply Lemmas 2 and 1 to the complements of $g_1 \cup \dots \cup g_p$ and $f_{11} \cup \dots \cup f_{1p}$ with respect to $g_1 \cup \dots \cup g_N \cup f_{11} \cup \dots \cup f_{1p}$). By rearranging indices we may now suppose that $f_{2j} \smile f_{1j} = g_j$ for $j = 1, \dots, r$ with $r \leq p$, that $f_{2j} \smile g_j$ for $j = p + 1, \dots, q$, and that $f_{2j} = 0$ for all other j . Then we may even suppose $f_{2j} = g_j$ for $j = p + 1, \dots, q$. Next, by changing the g_j with $j > q$ and increasing N if necessary, we

may suppose that each such g_j satisfies either $g_j \cap (f_1 \cup f_2) = 0$ or

$$g_j \leq \bigcup_{j=1}^r f_{2j};$$

letting g_{N+1} be a complement of

$$(g_1 \cup \dots \cup g_N) \cap \left(\bigcup_{j=1}^r f_{2j} \right)$$

with respect to

$$\bigcup_{j=1}^r f_{2j}$$

and writing N again for the former $N+1$ we may now suppose that

$$\bigcup_{j=1}^r f_{2j} \leq \bigcup_{j=q+1}^N g_j.$$

Then

$$\bigcup_{j=q+1}^N g_j = \bigcup_{j=1}^r f_{2j} \cup f_0$$

are two direct decompositions of the same element (with f_0 a suitable complement) and Lemma 6 applies. We derive direct decompositions for all elements used previously, such that (using the previous notation again) we may even suppose that $f_{2j} \sim g_{q+j}$ for $j = 1, \dots, r$. Now a direct decomposition for f is

$$\bigcup_{j=1}^N f_j$$

with $f_j = f_{1j}$ for $j = 1, \dots, p$, $f_j = f_{2j}$ for $j = p+1, \dots, q$, $f_{q+j} = f_{2j}$ for $j = 1, \dots, r$, and $f_j = 0$ for all other j . When the decompositions for f_1, f_2 used in (3.2) are replaced by this decomposition for f the number F_j is altered by -1 if $j = 1, \dots, r$ and by $+1$ if $j = q+1, \dots, q+r$. However, the equality of E_j, F_j can be restored as follows. For each fixed $j = 1, \dots, r$ we have $g_j \sim g_{q+j}$. If $F_j < 2 + F_{q+j}$ then there must be an $i > 2$ with $f_{ij} = 0$ and $f_{i,q+j} \sim g_{q+j}$; in this case we interchange these elements so that $f_{ij} \sim g_j$ and $f_{i,q+j} = 0$. If however $F_j \geq 2 + F_{q+j}$, then $E_j \geq 2 + E_{q+j}$ and there must be some i for which $e_{ij} \sim g_j$ and $e_{i,q+j} = 0$; in this case we interchange these two elements so that $e_{ij} = 0$ and $e_{i,q+j} \sim g_{q+j}$.

This completes the proof of (3.5).

Proof of (3.6). Suppose

$$(3.7) \quad e_1 + \dots + e_n + h_1 + \dots + h_p \sim f_1 + \dots + f_m + h_1 + \dots + h_p(d).$$

We wish to deduce $e_1 + \dots + e_n \sim f_1 + \dots + f_m(d)$. Proof by induction will apply here and we need only consider (3.7) with $p = 1$. Then, as detailed in (3.2), there are independent g_1, \dots, g_N and direct decompositions of the e_i, f_i , and h_1 into elements each of which is perspective to one of the g_j . We may replace h_1 in (3.7) by the lattice union of its corresponding set of g_j . We note

that h_1 may be assigned two different sets L and R of g_j according as h_1 appears on the left or right of (3.7). Since the two replacements for h_1 are perspective by Lemma 3, we may apply Lemmas 2, 1, and 6 to obtain decompositions of the g_j in L but not in R and of the g_j in R but not in L into new elements (which we will again call g_j) which are perspective in pairs. Thus we may suppose (3.7) given in the form

$$(3.8) \quad e_1 + \dots + e_n + g_1 + \dots + g_r \vee f_1 + \dots + f_m + g_{r+1} + \dots + g_{2r}(d)$$

with g_1, \dots, g_{2r} a subset of the g_1, \dots, g_N mentioned in (3.2) and with $g_i \vee g_{r+i}$ for $i = 1, \dots, r$.

For fixed j let E_j be the number of i for which $e_{ij} \vee g_j$ and let F_j be the number of i for which $f_{ij} \vee g_j$. Then for $j > 2r$ we deduce from (3.8) that $E_j = F_j$. If $j \leq r$ we obtain $E_j + 1 = F_j$, $E_{r+j} = F_{r+j} + 1$. Hence at least one of $E_j < E_{r+j}$, $F_j > F_{r+j}$ holds. If $E_j < E_{r+j}$ there must be an e_i for which $e_{ij} = 0$ and $e_{i,r+j} \vee g_{r+j}$; in that case we interchange these elements e_{ij} , $e_{i,r+j}$ so that now $e_{ij} \vee g_{r+j} \vee g_j$ and $e_{i,r+j} = 0$, thus obtaining $E_j = F_j$, $E_{r+j} = F_{r+j}$ for the new decompositions. In the same way, if $F_j > F_{r+j}$ we can rearrange the decomposition of some f_i to obtain $E_j = F_j$ and $E_{r+j} = F_{r+j}$. After this is done for each $j \leq r$ we obtain decompositions in terms of g_1, \dots, g_N for which (3.2) can be easily verified.

This completes the proof of Theorem 3.1.

COROLLARY TO THEOREM 3.1. *Two elements e, f in S satisfy $e + u = f + u$ for some u in G if and only if $e \vee f$.*

(It is easy to prove directly that $e = f$ if and only if $e = f$.)

Remark. The relation $x \equiv y$ in G can also be characterized in terms of decompositions in S but we omit the somewhat involved statement. In the special case of S a Boolean ring, $e \vee f$ holds if and only if $e = f$, and Theorem 3.1 shows that $x + u = y + u$ if and only if $x \equiv y$. Thus the cancellation law holds in G if S is a Boolean ring but not if S is a general relatively complemented modular lattice.

THEOREM 3.2. *$me + u = e_1 + \dots + e_n + u$ as in the condition (2.3) if and only if there are direct decompositions*

$$e_i = \bigcup_{j=1}^m e_{ij}' \quad (i = 1, \dots, n), \quad e = \bigcup_{j=1}^n e_{ij}'' \quad (j = 1, \dots, m)$$

with $e_{ij}' \vee e_{ij}''$ for all i, j .

Proof. Apply Theorem 3.1 with $f_1 = \dots = f_m = e$ to obtain the decompositions of (3.2) with g_1, \dots, g_N which we may suppose all non-zero. For given p, q let $J(p, q)$ be the set of j for which f_{pj} and e_{qj} are both different from zero, and the number of $r < p$ for which f_{rj} is different from zero is equal to the

number of $r < q$ for which $e_{r,j}$ is different from zero. Set

$$e_{p,q}'' = \bigcup f_{p,q}(j \in J(p, q))$$

$$e_{p,q}' = \bigcup e_{q,j}(j \in J(p, q)).$$

With this construction the theorem can be easily verified.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. G. Birkhoff, *Lattice theory*, 2nd ed. (New York, 1948).
3. I. Halperin, *On the transitivity of perspectivity in continuous geometries*, Trans. Amer. Math. Soc., vol. 44 (1938), 537-562.
4. G. G. Lorentz, *Multiply subadditive functions*, Can. J. Math., vol. 4 (1952), 455-462.
5. H. Nakano, *Modulated linear spaces*, J. Fac. Sci., Univ. Tokyo, Sec. I, vol. 6 (1951), 85-131.
6. J. von Neumann, *Continuous geometries*, Proc. Nat. Acad. Sci., vol. 22 (1936), 92-108.
7. ———, *Lectures on continuous geometry*, planographed (The Institute for Advanced Study, Princeton, 1935-1937).

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MAP-COLOUR THEOREMS

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Introduction. A map will be called *k*-chromatic or said to have *chromatic number k* if *k* is the least positive integer having the property that the countries of the map can be divided into *k* mutually disjoint (colour) classes in such a way that no two countries which have a common frontier line are in the same (colour) class. Heawood [4] proved that for $h > 1$ the chromatic number of a map on a surface of connectivity *h* is at most n_h , where

$$n_h = [\frac{1}{2}(7 + \sqrt{24h - 23})].$$

([*x*] denotes the integral part of *x*.) It is known also [5] that for $2 < h < 15$, n_h different colours are sometimes needed, because maps consisting of n_h mutually adjacent countries can be drawn on the surfaces concerned.

The main purpose of this paper is to establish the following:

THEOREM I. *For $h = 3$ and for $h > 5$ a map on a surface of connectivity *h* with chromatic number n_h always contains n_h mutually adjacent countries.*

It follows from this theorem that for $h = 3$ and for $h > 5$ every n_h -chromatic map on a surface of connectivity *h* consists essentially (in a sense which will become clear later) of n_h mutually adjacent countries; all maps drawn on such a surface which do not contain n_h mutually adjacent countries can be coloured with less than the full n_h colours.

On the other hand, it is possible that for some values of *h* the surfaces of connectivity *h* are such that all maps on them can be coloured using less than n_h colours. The theorem furnishes a procedure for deciding this. For any given surface of connectivity $h > 1$, n_h colours are needed only if a map consisting of n_h mutually adjacent countries can be drawn on the surface. (For $h = 2$ and 4 this follows not from Theorem I but from Theorems II and III.) For $h = 1$ ($n_h = 4$) the theorem is clearly not true, and it has of course never been proved, or disproved, that four colours are always sufficient to colour a map on the sphere or in the plane.

For the cases $h = 2$ and $h = 4$, Theorems II and III respectively will be proved; they are somewhat weaker than Theorem I.

The following table shows the values of n_h corresponding to values of *h* up to 16.

h : 1,2,3,4,5,6,7, 8, 9,10,11,12,13,14,15,16

n_h : 4,6,7,7,8,9,9,10,10,10,11,11,12,12,12,13.

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1. Combinatorial basis. It is convenient to consider graphs rather than maps, the graphs being the duals of maps and therefore such that no node is joined to itself by an edge, two nodes are joined by at most one edge, two edges meet in a node or not at all, and there is no crossing of edges. A graph will be called k -chromatic, or said to have chromatic number k , if k colours are just sufficient to colour the nodes in such a way that two nodes which are joined by an edge are never coloured the same. Theorem I, which concerns maps, is equivalent to the following theorem concerning graphs:

THEOREM I'. For $h = 3$ and for $h > 5$, a graph of chromatic number n_h on a surface of connectivity h always contains n_h mutually adjacent nodes.

To prove this theorem some combinatorial notations and results are necessary.

If Γ is a graph, its chromatic number will be denoted by $K(\Gamma)$. A graph will be called *critical* if it has no subgraph of smaller order with the same chromatic number. Clearly, if Γ is critical the degree of every node of Γ is at least $K(\Gamma) - 1$. De Bruijn proved [2] that an infinite k -chromatic graph always contains a finite k -chromatic subgraph. It follows that a k -chromatic graph always contains a critical k -chromatic subgraph, and a critical graph is finite and connected. If N denotes the number of nodes and E denotes the number of edges of a critical k -chromatic graph, since the degree of every node is at least $k-1$, the following simple inequality holds:

$$1.1 \quad \frac{2E}{N} > k - 1.$$

This inequality was improved by Brooks [1] who proved that if $k \geq 4$ a k -chromatic graph either contains k nodes such that every pair are joined by an edge, or it contains a node of degree $> k$. Hence

$$1.2 \quad \text{if } k \geq 4 \text{ and } N > k \text{ then } \frac{2E}{N} > k - 1.$$

(A graph which consists of n nodes, every pair of distinct nodes being joined by an edge, is usually called a *complete n -graph*. This terminology will be adopted.)

It is necessary for the purpose of this paper to strengthen 1.2 considerably for $N \leq k + 3$, and the following will be now established:

1.3 A k -chromatic graph which does not contain a complete k -graph as a sub-graph contains at least $k + 2$ nodes.

1.4 In the notation of 1.2, if $k \geq 5$ and $N = k + 2$ for a critical graph, then

$$\frac{2E}{N} > k + 1 - \frac{12}{k + 2}.$$

1.5 In the same notation, if $k \geq 5$ and $N = k + 3$ for a critical graph, then

$$\frac{2E}{N} > k + 2 - \frac{24}{k + 3}.$$

In a previous paper [3] I proved the following result:

1.6 If $0 \leq n \leq k-1$, a k -chromatic graph either contains a complete $(k-n)$ -graph as a subgraph or it has at least $k+n+2$ nodes.

1.3 follows from this on substituting $n=0$.

Proof of 1.4. It also follows that a critical k -chromatic graph of order $k+2$ contains a complete $(k-1)$ -graph as a subgraph. Let $k \geq 5$ and let the nodes of such a graph be denoted by $a_1, a_2, \dots, a_{k-1}, b_1, b_2, b_3$; where every pair of a_1, a_2, \dots, a_{k-1} , is joined by an edge. Because the graph is critical, b_1, b_2 , and b_3 are each joined to at least $k-1$ nodes. The number of edges in the graph, consistent with this requirement, is least (i.e., the most economical distribution of edges is obtained) if each of b_1, b_2 , and b_3 is joined to the other two and to $k-3$ of the nodes a_1, a_2, \dots, a_{k-1} . In this case the graph contains

$$\frac{1}{2}(k-1)(k-2) + 3(k-3) + 3$$

edges; with any other distribution of edges it contains more. Hence for such a graph,

$$\frac{2E}{N} > k+1 - \frac{12}{k+2}.$$

But with the distribution described above, it is easy to see that unless b_1, b_2 , and b_3 are all joined to the same $k-3$ nodes from among a_1, a_2, \dots, a_{k-1} , the chromatic number of the graph is $k-1$ ($k \geq 5$ was assumed). If b_1, b_2 , and b_3 are all joined to the same $k-3$ nodes from among a_1, a_2, \dots, a_{k-1} , then these nodes together with b_1, b_2 , and b_3 form a set of k nodes of which each pair is joined by an edge, so that the graph contains a complete k -graph as a subgraph and is therefore not critical. This most economical distribution is therefore not permissible, and so for a critical k -chromatic graph of order $k+2$,

$$\frac{2E}{N} > k+1 - \frac{12}{k+2},$$

and this proves 1.4.

Proof of 1.5. By 1.6 a critical k -chromatic graph of order $k+3$ contains a complete $(k-2)$ -graph as a subgraph. It is easiest to prove 1.5 by considering two alternatives: the graph contains a complete $(k-1)$ -graph as a subgraph or it does not.

(i) *The graph contains a complete $(k-1)$ -graph as a subgraph.* Let the nodes of the graph be denoted by $a_1, a_2, \dots, a_{k-1}, b_1, b_2, b_3, b_4$; where every pair of a_1, a_2, \dots, a_{k-1} is joined by an edge. Because the graph is critical, b_1, b_2, b_3 , and b_4 are each joined to at least $k-1$ nodes. The number of edges in the graph, consistent with this requirement, is least (i.e., the most economical distribution of edges is obtained) if each of b_1, b_2, b_3 , and b_4 is joined to the other three and to $k-4$ of the nodes a_1, a_2, \dots, a_{k-1} . In this case the graph contains

$$\frac{1}{2}(k-1)(k-2) + 4(k-4) + 6$$

edges; with any other distribution of edges it contains more. Hence for such a graph,

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3}.$$

But with the distribution described above it is easy to see that unless b_1, b_2, b_3 , and b_4 are all joined to the same $k-4$ nodes from among a_1, a_2, \dots, a_{k-1} , the chromatic number of the graph is $k-1$ ($k \geq 5$ was assumed). If b_1, b_2, b_3 , and b_4 are all joined to the same $k-4$ nodes from among a_1, a_2, \dots, a_{k-1} then these nodes together with b_1, b_2, b_3 , and b_4 form a set of k nodes of which each pair is joined by an edge, so that the graph contains a complete k -graph as a subgraph and is therefore not critical. This most economical distribution is therefore not permissible, and so, for a critical k -chromatic graph of order $k+3$ containing a complete $(k-1)$ -graph as a subgraph,

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3}.$$

(ii) *The graph does not contain a complete $(k-1)$ -graph as a subgraph.* Let Γ denote the graph. It contains a complete $(k-2)$ -graph as a subgraph and is critical (by 1.6 with $n=2$). Let a be any node of Γ which does not belong to this complete $(k-2)$ -graph and let Γ' denote the graph obtained from Γ by deleting a and all edges incident in a . Γ' is $(k-1)$ -chromatic and it is not critical. For if Γ' were critical, then a would have to be joined to every node of Γ' , since Γ is k -chromatic; and as Γ' contains a complete $(k-2)$ -graph as a subgraph, Γ would contain a complete $(k-1)$ -graph as a subgraph, contrary to hypothesis. Therefore, Γ' contains a node b such that the graph obtained from Γ' by deleting b and all edges incident in b , say Γ'' , is $(k-1)$ -chromatic.

Now Γ'' is $(k-1)$ -chromatic and of order $k+1$ and does not contain a complete $(k-1)$ -graph as a subgraph. It is therefore critical: if it were not, it would contain a $(k-1)$ -chromatic subgraph of order k without a complete $(k-1)$ -graph, whereas, by 1.6 with $k-1$ in place of k , and $n=0$, a $(k-1)$ -chromatic graph either contains a complete $(k-1)$ -graph as a subgraph or it has at least $k+1$ nodes. Since Γ'' is critical, by 1.4 with $k-1$ in place of k , the number of its edges is at least $\frac{1}{2}k(k+1) - 5$.

The nodes a and b in Γ are each of degree $\geq k-1$, since Γ is critical, and so contribute at least $2(k-1) - 1$ edges to Γ . The number of edges in Γ is therefore at least

$$\frac{1}{2}k(k+1) - 5 + 2(k-1) - 1 = \frac{1}{2}k^2 + \frac{5}{2}k - 8.$$

Hence, for Γ :

$$\frac{2E}{N} > k + 2 - \frac{22}{k+3}.$$

Under assumption (i) it was shown that

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3},$$

so that 1.5 is now proved.

Theorems 1.2, 1.3, 1.4, and 1.5 form the combinatorial basis for Theorem I.

2. Topological basis and the proof of Heawood's formula. Let a graph be drawn on a surface of connectivity h which divides the surface into polygonal regions. If N denotes the number of nodes, E the number of edges, and F the number of polygonal regions into which the surface is divided, then Euler's Theorem states that

$$N + F - E = 3 - h.$$

If there are regions on the surface which are bounded by more than three edges, it is possible to add new edges until a graph is obtained which divides the surface into polygons bounded by three edges, i.e., triangles. (It is to be understood of course that we speak of polygons and triangles drawn on the surface in question, whose vertices are nodes and whose sides are edges of the graph.) The number of nodes of the new graph is still N . Let the number of edges be E' and the number of triangular regions be F' , then $E' \geq E$ and $F' \geq F$. Now every triangle is bounded by three edges and every edge separates two triangles, hence

$$3F' = 2E'.$$

By Euler's Theorem,

$$3N + 3F' - 3E' = 9 - 3h;$$

hence

$$3N - E' = 9 - 3h,$$

and so

$$\frac{2E'}{N} = 6 \left(1 + \frac{h-3}{N} \right).$$

Since $E' \geq E$, for the original graph,

$$2.1 \quad \frac{2E}{N} < 6 \left(1 + \frac{h-3}{N} \right).$$

A graph drawn on the surface which does not divide it into polygonal regions can be drawn on a surface with smaller connectivity, or can be made to divide the surface into polygonal regions by the addition of edges. Thus 2.1 holds for all graphs drawn on a surface of connectivity h .

Let k be the chromatic number of a graph drawn on a surface of connectivity h . Such a graph contains a critical k -chromatic subgraph. Let this subgraph have N nodes and E edges. Then clearly $N \geq k$, and the degree of each node is at least $k-1$, so that $2E/N \geq k-1$. Hence, from 2.1, if $h \geq 3$,

$$k-1 < 6 \left(1 + \frac{h-3}{k} \right).$$

It follows that:

If $h \geq 3$, every graph drawn on a surface of connectivity h can be coloured using

at most n_h colours, where n_h is the greatest integer satisfying

$$2.2 \quad n_h - 1 < 6 \left(1 + \frac{h-3}{n_h} \right).$$

If $h = 2$ we have from 2.1 that $k - 1 < 6$, that is, $k \leq 6$.

The value of n_h from 2.2 explicitly is

$$\left[\frac{1}{2} (7 + \sqrt{24h - 23}) \right]$$

and, when $h = 2$, gives the correct value 6. Thus we have a very simple proof of the well-known result quoted in the beginning of this paper.

3. Proof of Theorem I. To prove Theorem I it is to be shown that for $h = 3$ and $h \geq 5$ the only critical n_h -chromatic graph which can be drawn on a surface of connectivity h is the complete n_h -graph. To do this will first be proved that for $h = 3$ and $h \geq 5$ no critical n_h -chromatic graph of order $\geq n_h + 4$ can be drawn on a surface of connectivity h . Then it will be proved that no critical n_h -chromatic graph of order $n_h + 2$ or $n_h + 3$ can be drawn on such a surface. Theorem I will then follow by 1.3.

Suppose a critical n_h -chromatic graph of order $\geq n_h + 4$ is drawn on a surface of connectivity h . If E denotes the number of edges and N the number of nodes then by 1.2,

$$\frac{2E}{N} > n_h - 1;$$

and by 2.1, since $h \geq 3$,

$$\frac{2E}{N} < 6 \left(1 + \frac{h-3}{n_h+4} \right).$$

Hence

$$3.1 \quad n_h - 1 < 6 \left(1 + \frac{h-3}{n_h+4} \right),$$

while n_h satisfies the inequalities:

$$3.2 \quad n_h - 1 < 6 \left(1 + \frac{h-3}{n_h} \right),$$

$$3.3 \quad n_h > 6 \left(1 + \frac{h-3}{n_h+1} \right).$$

From 3.1,

$$n_h^2 - 3n_h < 6h + 9,$$

and from 3.3,

$$n_h^2 - 5n_h > 6h - 11;$$

hence $2n_h < 20$, that is, $n_h < 10$. It remains to examine those cases where $n_h < 10$.

Case $n_h = 7$. By the table on page 480, the case to be examined is $h = 3$. (The

case $h = 4$ is excluded from Theorem I.) By 1.2 with $k = 7$,

$$\frac{2E}{N} > 6;$$

and by 2.1 with $h = 3$,

$$\frac{2E}{N} < 6.$$

This is a contradiction. Actually, 1.2 with $k = 7$ applies to all critical 7-chromatic graphs of order exceeding 7, and so we have completed the proof of Theorem I for the case $h = 3$. (The theorem with $h = 3$, was first established by P. Ungár. His proof is different from this one.)

Case $n_h = 8$. From the table, if $n_h = 8$ then $h = 5$. By 1.2 with $k = 8$,

$$\frac{2E}{N} > 7.$$

By 2.1 with $h = 5$ and $N > n_h + 4 = 12$,

$$\frac{2E}{N} < 6\left(1 + \frac{2}{12}\right) = 7.$$

This is a contradiction.

Case $n_h = 9$. From the table, if $n_h = 9$ then $h = 6$ or $h = 7$. Consider first the case $h = 7$. By 1.2 with $k = 9$,

$$\frac{2E}{N} > 8.$$

By 2.1 with $h = 7$ and $N > n_h + 4 = 13$,

$$\frac{2E}{N} < 6\left(1 + \frac{4}{13}\right) = 7\frac{11}{13}.$$

This is a contradiction. *A fortiori* the case $h = 6$ would lead to a contradiction.

Case $n_h = 10$. From the table, if $n_h = 10$ then $h = 8$ or $h = 9$ or $h = 10$. Consider first the case $h = 10$. By 1.2 with $k = 10$,

$$\frac{2E}{N} > 9.$$

By 2.1 with $h = 10$ and $N > n_h + 4 = 14$,

$$\frac{2E}{N} < 6\left(1 + \frac{7}{14}\right) = 9.$$

This is a contradiction. *A fortiori* the cases $h = 8$ and $h = 9$ lead to a contradiction.

These contradictions prove that for $h \geq 5$ no critical n_h -chromatic graph of order $> n_h + 4$ can be drawn on a surface of connectivity h , and that for $h = 3$ the only critical n_h -chromatic graphs which can be drawn on a surface of connectivity h are the complete n_h -graphs.

It remains to see whether an n_h -chromatic critical graph of order $n_h + 2$ or $n_h + 3$ can be drawn on a surface of connectivity h . These cases will be considered in turn:

Graphs of order $n_h + 2$. Suppose a critical n_h -chromatic graph of order $n_h + 2$ is drawn on a surface of connectivity h . By 1.4 for such a graph,

$$\frac{2E}{N} > n_h + 1 - \frac{12}{n_h + 2}.$$

By 2.1,

$$\frac{2E}{N} < 6\left(1 + \frac{h-3}{n_h+2}\right);$$

hence

$$n_h + 1 - \frac{12}{n_h + 2} < 6\left(1 + \frac{h-3}{n_h+2}\right),$$

that is, $n_h^2 - 3n_h < 6h + 3$. But from the definition of n_h , for $h > 3$,

$$n_h > 6\left(1 + \frac{h-3}{n_h+1}\right),$$

that is, $n_h^2 - 5n_h > 6h - 11$, and so $2n_h < 14$, or $n_h < 7$. But we have already disposed of the case $n_h = 7$ ($h = 3$).

Graphs of order $n_h + 3$. Suppose a critical n_h -chromatic graph of order $n_h + 3$ is drawn on a surface of connectivity h . By 1.5, for such a graph,

$$\frac{2E}{N} > n_h + 2 - \frac{24}{n_h + 3}.$$

By 2.1,

$$\frac{2E}{N} < 6\left(1 + \frac{h-3}{n_h+3}\right);$$

hence

$$n_h + 2 - \frac{24}{n_h + 3} < 6\left(1 + \frac{h-3}{n_h+3}\right),$$

that is, $n_h^2 - n_h < 6h + 17$. But by the definition of n_h ,

$$n_h > 6\left(1 + \frac{h-3}{n_h+1}\right),$$

that is, $n_h^2 - 5n_h > 6h - 11$, and so $4n_h < 28$, or $n_h < 7$. But we have already disposed of this case. This completes the proof of Theorem I.

The cases $h = 2$ and $h = 4$ have not been included. If $h = 2$ then $n_h = 6$, and the graphs drawn on these surfaces must be such that

$$\frac{2E}{N} < 6\left(1 - \frac{1}{N}\right).$$

The theorem of Brooks (1.2 above) states only that for critical 6-chromatic

graphs of order > 6 , $2E/N > 5$; it is not strong enough to settle the question. It is of course well known that a map consisting of six mutually adjacent countries can be drawn on the projective plane [5] (for which $h = 2$); but I do not know whether it is possible to draw a map on a surface of connectivity 2 which does not contain six mutually adjacent countries and is nevertheless 6-chromatic. If $h = 4$ then $n_h = 7$ and the graphs drawn on these surfaces must be such that

$$\frac{2E}{N} < 6\left(1 + \frac{1}{N}\right).$$

Theorem 1.2 states only that for critical 7-chromatic graphs of order exceeding 7, $2E/N > 6$; so that it fails to deal with this case also. I think that it is very unlikely that a 7-chromatic map which does not contain seven mutually adjacent countries can be drawn on a surface of connectivity 4.

4. Weaker theorems for $h = 2$ and for $h = 4$. It is curious, but not unusual in this subject, that the surfaces of greater connectivity should be more easily amenable to treatment than the simpler surfaces with low connectivity. But instead in the cases $h = 2$ and $h = 4$, I will prove the following:

THEOREM II. *A 6-chromatic map on a surface of connectivity 2 either contains 6 mutually adjacent countries, or a map containing 6 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines and uniting those countries which they separate.*

THEOREM III. *A 7-chromatic map on a surface of connectivity 4 either contains 7 mutually adjacent countries, or a map containing 7 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines and uniting those countries which they separate.*

The proof is based on the following simple

LEMMA. *If $h > 2$, a map on a surface of connectivity h contains a country with fewer than n_h neighbours.*

A graph in which the degree of every node is at least n_h has at least $n_h + 1$ nodes. If N denotes the number of nodes and E the number of edges of such a graph then

$$\frac{2E}{N} > n_h.$$

On the other hand, if the graph can be drawn on a surface of connectivity h , by 2.1,

$$\frac{2E}{N} < 6\left(1 + \frac{h-3}{N}\right).$$

Hence

$$n_h < 6\left(1 + \frac{h-3}{n_h+1}\right) \quad \text{if } h > 2 \quad \text{and} \quad n_h < 6 \quad \text{if } h = 2.$$

But from the definition of n_h ,

$$n_h > 6 \left(1 + \frac{h-3}{n_h+1} \right) \quad \text{if } h > 2 \quad \text{and} \quad n_h = 6 \quad \text{if } h = 2.$$

So a graph on a surface of connectivity h contains a node with fewer than n_h neighbours. The Lemma follows.

Proof of Theorem II. The theorem is certainly true for maps containing 6 countries. We shall assume it to be true for maps containing not more than $C-1$ countries ($C \geq 7$) and deduce that it is true for maps containing C countries. The truth of the theorem will then follow by the induction principle.

Let M be a 6-chromatic map containing C countries, B frontier lines, and A frontier points common to three or more countries, and not having 6 mutually adjacent countries. If, on deleting a country from M , there remains a 6-chromatic map, then by the induction hypothesis the theorem is true for M . We may therefore suppose that on deleting any country from M there remains a 5-chromatic map, and in this case *every country has at least 5 neighbours having a common frontier line with it.*

By the Lemma, M therefore contains a country with just 5 neighbours.

Let X be such a country and let its neighbours be Y, Z, U, V, W . If each pair of Y, Z, U, V, W were neighbours then M would contain the 6 mutually adjacent countries X, Y, Z, U, V, W , contrary to hypothesis. So among Y, Z, U, V, W there are two countries which are not neighbours, say Y and Z . Let M' denote the map obtained from M by deleting the frontier line separating X and Y and the frontier line separating X and Z and uniting the three countries X, Y and Z into one country X' .

Then M' is 6-chromatic; for the chromatic number of M' is at most 6. Suppose it could be coloured using five colours. Then the map $M - X$, obtained from M by deleting X , could be coloured with five colours in such a way that Y and Z receive the same colour. In this colouring the countries Y, Z, U, V, W between them receive at most four colours. If X is now re-introduced into $M - X$, it can be given the fifth colour, and this gives a colouring of M using five colours, which contradicts the datum that M is 6-chromatic.

So M' is 6-chromatic and contains fewer countries than M , and therefore by the induction hypothesis it either contains 6 mutually adjacent countries, or a map containing 6 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines, and uniting those countries which they separate. *A fortiori* the same is true of M , and so the theorem is proved.

Proof of Theorem III. Similar to the proof of Theorem II, with $n_h = 7$ instead of 6.

It follows that a 7-chromatic map can be drawn on a surface of connectivity 4 if and only if there is room for a complete 7-graph on the surface.

NOTE. By a very similar method a short proof of the five-colour theorem of

Heawood [4] can be obtained. For a plane or spherical graph ($h = 1$), $2E/N < 6$; hence a map on the plane or the sphere contains a country with not more than five neighbours, of which two have no common frontier line. If the map obtained by uniting the country and two non-adjacent neighbours can be coloured using five colours, so can the original map. The five-colour theorem now follows by induction.

REFERENCES

1. R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc., vol. 37 (1941), 194.
2. N. G. de Bruijn and P. Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Proc. K. Nederl. Akad. Wetensch. Amsterdam, ser. A, vol. 54 (1951), 371.
3. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3), vol. 2 (1952), 69.
4. P. J. Heawood, *Map colour theorem*, Quarterly J. Math., vol. 24 (1890), 332.
5. For $h = 3, 5, 7, 9, 11, 13, 15$ see L. Heffter, *Über das Problem der Nachbargebiete*, Math. Ann., vol. 38 (1891), 477.
 For $h = 2$ see H. Tietze, *Einige Bemerkungen über das Problem des Kartenfärbens auf einseitigen Flächen*, Jber. dtsch. MatVer., vol. 19 (1910), 155; D. Hilbert and S. Cohn-Vossen, *Anschauliche Geometrie* (Berlin, 1932), translated as *Geometry and the imagination* (New York, 1952); W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical recreations and essays* (London, 1947).
 For $h = 4$ see I. N. Kagno, *A note on the Heawood colour formula*, J. Math. Phys., vol. 14 (1935), 228.
 For $h = 6$ see H. S. M. Coxeter, *The map colouring of unorientable surfaces*, Duke Math. J., vol. 10 (1943), 293.
 For $h = 8$ see R. C. Bose, *On the construction of balanced incomplete block designs*, Annals of Eugenics, vol. 9 (1939), 353.
 The cases $h = 10, 12, 14$: The connectivities of the surfaces obtained from the sphere and from the projective plane by attaching n handles are $2n + 1$ and $2n + 2$ respectively. Any map drawn on the surface of a sphere with n handles attached can also be drawn on a projective plane with n handles attached. It follows that $k_{2n+2} \geq k_{2n+1}$. Hence, by Heffter's results quoted above and by the table on p. 480, Heawood's result is best possible also for $h = 10, 12$, and 14 .

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ON THE SUBALGEBRAS OF FINITE DIVISION ALGEBRAS

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1. Introduction. In 1893 it was shown by Moore that the only commutative, associative division algebras with a finite number of elements are the well-known Galois fields [10, p. 220]. Twelve years later it was shown by Wedderburn that every associative division algebra with a finite number of elements is commutative [11], and hence a Galois field. It is conceivable that these results, particularly the theorem of Moore, motivated some of the work done by Dickson and published in two papers in 1906 [4;5]. The work referred to is an attempt to determine all commutative, non-associative¹ division algebras with a finite number of elements. The most complete result of Dickson states that there are only two commutative division algebras with unit element of order 3 over a Galois field $GF(q^k)$. One of these is the associative algebra $GF(q^{3k})$ and the other an algebra in which the multiplication is not associative. Since this non-associative algebra is discussed briefly in §4 the details will be omitted here. The methods used by Dickson in this connection are not capable of immediate generalization, and the problem of determining all commutative division algebras of order n over a finite field is still unsolved. Although Dickson apparently abandoned the problem shortly after the publication of the papers referred to above, his work in this connection should not be taken too lightly. It has been conjectured that this work may have led Dickson to his important discovery of cyclic algebras.

Before discussing the results contained in the present paper, it is desirable to make several definitions and some obvious remarks concerning finite division rings and finite division algebras.

A set G of elements is called a *quasigroup* with respect to a binary operation (\cdot) , if and only if:

- (i) $x \cdot y$ is uniquely determined for each ordered pair $x, y \in G$,
- (ii) the equations $a \cdot x = b$, $y \cdot a = b$ have unique solutions for each ordered pair $a, b \in G$.

A quasigroup with unit element is called a *loop*.

A set A of elements is called a *division ring* with respect to two binary operations, $(+)$ and (\cdot) , defined on A , if and only if:

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¹Here, non-associative algebra means one which is not necessarily associative. In the remainder of the paper, however, non-associative algebra will mean an algebra in which the multiplication is actually not associative.

- (i) the elements of A are an abelian group under $(+)$,
- (ii) the non-zero elements of A are a quasigroup under (\cdot) ,
- (iii) the two distributive laws $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ hold for all $x, y, z \in A$.

A division ring A is called a *division algebra* of order n over a field F , if and only if:

- (i) the additive group of A is a linear vector space of order n over F ,
- (ii) $a(a \cdot b) = a \cdot (ab) = (aa) \cdot b$ for all $a \in F$ and $a, b \in A$.

Note that if a division algebra A contains a unit element e , then the set of all multiples, $ae, a \in F$, is a subalgebra of A isomorphic to F . There is no loss of generality in denoting ae by a .

Let A be a division ring, and a any non-zero element of A . Then the mappings $x \rightarrow xa$ and $x \rightarrow ax$ are clearly non-singular mappings of A upon A . These mappings are denoted by $R(a)$ and $L(a)$ respectively. Let a, b be any two non-zero elements of A , and consider the system A_0 , consisting of the same elements as A , with multiplication defined by $x \cdot y = xR(a)^{-1} \cdot yL(b)^{-1}$. Denote by e the product $b \cdot a$, then clearly $e \cdot y = y \cdot e = y$ for all $y \in A_0$. Thus e is a unit element of A_0 ; furthermore, it is easily seen that A_0 is a division ring, and that it is commutative if A is commutative and $a = b$. It is seen then that the study of division rings is reduced to a study of division rings with unit element.

Now, if A is a finite division ring with unit e_1 , then clearly A contains a subring M isomorphic to a finite prime field $GF(p)$ for some rational prime p . By finite induction it may be shown that A has a basis e_1, \dots, e_n with respect to M . Thus a study of finite division rings is reduced to a study of division algebras with unit element over a Galois field.

In §2 finite commutative division algebras with unit element are studied. It is shown that every such algebra of even order contains a subalgebra of order 2, and that no such algebra of odd order contains a subalgebra of order 2. A well-known, and rather elementary, result in the theory of associative division algebras states that the order of every subalgebra of a division algebra is a divisor of the order of the algebra. Whether or not this result is valid for non-associative division algebras is not known. One application of the results obtained in §2 gives a little information in this connection in the case where the division algebra is a finite commutative algebra with unit element.

An attempt is made in §3 to determine, by the use of Theorem 1, all finite, commutative division algebras with unit element of order 4 over a Galois field. A theorem due to Dickson is sharpened somewhat, but not enough to solve the problem completely. In fact, Theorem 1 does not seem to give enough information to solve the weaker problem of determining all finite, commutative division algebras of order 4 whose automorphism group relative to the base field contains the cyclic group of order 4. It is possible, however, to find out something about the subalgebras of finite division algebras of order n whose automorphism group relative to the base field contains the cyclic group of order n . This is done in

§6, after it has been shown in §5 that there exist finite algebras of this kind which are not associative.

2. Finite commutative division algebras. Before proceeding to the main theorem of this section, it is necessary to make some remarks concerning normality of subloops of a given loop L . The subloop G of L is said to be *normal* in L if and only if the following condition is satisfied [2, p. 256]: for arbitrary $x, y \in L$, if in the equation $(xy)g_1 = (xg_2)(yg_3)$, any two of the elements g_1, g_2, g_3 , are arbitrary elements of G , then the third is a uniquely determined element of G .

Let A be a division algebra over a field F . Denote by L the set of non-zero elements of A , and by G the set of non-zero elements of F . It is clear that G is a normal subloop of L and that the set of all distinct cosets $xG, x \in L$, is a loop, called the quotient loop of L modulo G , and denoted by L/G .

The following lemma is due to Griffin [8, p. 728].

LEMMA 1. *If Q is a finite, commutative quasigroup with an even number of elements, then the equation $y^2 = a$ has an even number of distinct solutions for each $a \in Q$.*

The proof, which will be omitted, follows from a consideration of the main diagonal of the multiplication table of Q .

LEMMA 2. *If A is a commutative division algebra of order $2n$ over a field $F = GF(q^k)$, $q > 2$, and if L and G denote the non-zero elements of A and F respectively, then in the quotient loop L/G the equation $X^2 = C$ has either two distinct solutions or no solution.*

Proof. First note that the loop L/G contains

$$t = 1 + q^k + q^{2k} + \dots + q^{(2n-1)k}$$

elements. Clearly t is even, and hence by Lemma 1, the equation $X^2 = C$ has an even number of distinct solutions in L/G . Suppose that for some $C = cG \in L/G$ the equation $X^2 = C$ has more than two distinct solutions. Then there exist at least three distinct elements $xG, yG, zG \in L/G$ such that $(xG)^2 = (yG)^2 = (zG)^2$. These imply $x^2G = y^2G = z^2G$, and hence $x^2 = \alpha y^2 = \beta z^2$, where α, β are non-squares in F . Thus $y^2 = \alpha^{-1}\beta z^2$, or since $\alpha^{-1}\beta = \gamma^2$ in F , $y^2 = \gamma^2 z^2$, which implies that $y = \pm \gamma z$, or $yG = zG$, a contradiction.

COROLLARY. *Exactly half of the equations $X^2 = C, C \in L/G$ have solutions in L/G .*

THEOREM 1. *If A is a commutative division algebra with unit element of order $2n$ over a field $F = GF(q^k)$, $q > 2$, then A contains a unique subalgebra M of order 2 over F . The subalgebra M is isomorphic to the field $GF(q^{2k})$, and may be characterized as the set of all elements of A which satisfy quadratic equations with coefficients in F .*

Proof. Denote by 1 the unit element of A . Then, with the notation of Lemma 2, since the equation $X^2 = 1G$ has a solution $X = 1G$ in L/G , it follows that the equation has exactly two distinct solutions. Let $X = eG$ be the second solution. It is clear that 1, e are linearly independent over F , and that $e^2 = \phi$, where ϕ is a non-square in F . Thus the subspace spanned by 1 and e is a subalgebra M of A . Denote by ρ a zero of the polynomial $\lambda^2 - \phi$, irreducible in $F[\lambda]$; it is easily seen that M is isomorphic to the field $F(\rho) = GF(q^{2k})$ under the correspondence $\alpha + \beta e \leftrightarrow \alpha + \beta \rho$. A contains no other subalgebra of order 2 over F . This follows from the fact that $1G$ and eG are the only solutions of $X^2 = 1G$ in L/G . Clearly, every element of M satisfies a quadratic equation with coefficients in F . Let S denote the set of all such elements, so that $M \subseteq S$. If $x \in S$, and x is a scalar, then $x \in M$, since $F \subset M$. Let $x \in S$, and assume that x is not a scalar. Then $x^2 = \alpha x + \beta$, for some $\alpha, \beta \in F$, and

$$(x - \frac{1}{2}\alpha)^2 = \beta + (\frac{1}{4}\alpha)^2.$$

Thus $(x - \frac{1}{2}\alpha)G$ satisfies $X^2 = 1G$ in L/G , and it follows that $(x - \frac{1}{2}\alpha)G = eG$ since $x \notin G$. Hence $x - \frac{1}{2}\alpha = \gamma e$ for some $\gamma \in G$, and $x = \frac{1}{2}\alpha + \gamma e \in M$. Clearly then $S \subseteq M$, which together with $M \subseteq S$ implies that $S = M$. This completes the proof of the theorem.

In a finite commutative loop of odd order the equation $x^2 = a$ has a unique solution for each a (as in Lemma 1, consider the main diagonal of the multiplication table). It follows that if A is a commutative division algebra with unit element of odd order over a field $GF(q^k)$, $q > 2$, then $x^2 = a \in F$ implies that $x \in F$. Thus, if G, C are the sets of non-zero elements of A and F respectively, then the loop G/C is of odd order. Hence the equation $X^2 = C$ has the unique solution $X = C$. Let $x^2 = a \in C$ for some $x \in G$, then $(xG)^2 = C$, which implies that $xG = C$, or $x \in C \subset F$. Now, $x^2 \in F$ implies $x \in F$ is equivalent to saying that A contains no subalgebra of order two over F . This, together with Theorem 1, implies that no finite commutative division algebra with unit of odd order over a field $GF(q^k)$, $q > 2$, can contain a subalgebra of even order.

3. Finite commutative division algebras of order 4. Theorem 1 may be used to sharpen somewhat a theorem of Dickson [4, p. 381]. This will be accomplished by proving Lemma 4, from which the desired result readily follows. The proof of Lemma 3 follows immediately from the observation that in any commutative division ring $b^3 = a^2$ implies $b = \pm a$.

LEMMA 3. *Let A be a commutative division ring and M any proper subring of A . If b is an element of A such that $b^2 \in M$, then b^2 is a non-square in M if and only if $b \in A - M$.*

LEMMA 4. *If A is a commutative division algebra with unit of order 4 over a finite field $F = GF(q^k)$, $q > 2$, and if M is the unique subalgebra of order 2 over F described in Theorem 1, then every element of M is the square of some element of A .*

¹ $A - M$ means the usual set-theoretic complement of M in A .

Proof. Let $1, e$ be a basis for M . Then, if $x \in A - M$, $1, e, x, ex$, are a basis for A . Thus, every $x \in A - M$ satisfies a quadratic equation with coefficients in M , for if $x \in A - M$, then there exist in F four elements a_i ($i = 0, 1, 2, 3$) such that $x^2 = a_0 + a_1e + a_2x + a_3ex = \psi + \mu x$, where $\psi, \mu \in M$. Define y by $y = x - (\frac{1}{2}\mu)$, then $y \in A - M$ and $y^2 = \psi + (\frac{1}{2}\mu)^2 \in M$. Let $v = \psi + (\frac{1}{2}\mu)^2$, so that $y^2 = v$. By Lemma 3, v is a non-square in M . Since yG satisfies the equation $X^2 = vG$ in L/G , it follows from Lemma 2 that there exists a second solution $X = zG \neq yG$. Clearly, $z^2 = \gamma y^2 = \gamma v$, where γ is a non-square in F . By Theorem 1, M is isomorphic to $GF(q^{2k})$, and since $GF(q^{2k})$ is the root field of the polynomial $\lambda^2 - \gamma$, it follows that γ is the square of some element of M . Thus, $z^2 = \gamma v$ is a non-square in M , and hence $z \in A - M$ by Lemma 3.

Suppose that $\beta_0 + \beta_1e + \beta_2y + \beta_3z = 0$, where the $\beta_i \in F$. Then

$$(\beta_0 + \beta_1e)^2 + 2\beta_2(\beta_0 + \beta_1e)y + \beta_2^2y^2 = \beta_3^2z^2,$$

and since $y^2, z^2 \in M$, this equation implies that $\beta_2(\beta_0 + \beta_1e)y \in M$. Thus, $\beta_2(\beta_0 + \beta_1e) = 0$, for otherwise $y \in M$, a contradiction. If $\beta_2 = 0$, so that $\beta_0 + \beta_1e + \beta_3z = 0$, then, since $z \in A - M$, it follows that each of the remaining β_i is zero. However, if $\beta_0 + \beta_1e = 0$, so that $\beta_2y + \beta_3z = 0$, then $\beta_2 = \beta_3 = 0$, for otherwise $yG = zG$ in L/G , a contradiction. It is seen then that $1, e, y, z$, are linearly independent over F .

Assume that the elements of G have been ordered in some way and let η_i denote the i th element in this ordering. Then for each $i = 1, \dots, q^k - 1$, define y_i by $y_i = y + \eta_i z$. Clearly each $y_i \in A - M$, and hence there exist $\mu_i \in M$ such that $(y_i - \mu_i)^2 = v_i \in M$ for $i = 1, \dots, q^k - 1$. Let $y'_i = y_i - \mu_i$, and note that each $y'_i \in A - M$. It is easily verified that the $q^k + 1$ cosets y'_iG, yG , and zG are distinct. It follows that the $q^{2k} - 1$ elements of A contained in these cosets are distinct elements of $A - M$. Denote these elements by b_j , ($j = 1, \dots, q^{2k} - 1$). By Lemma 3, each b_j^2 is a non-square in M . Furthermore, it is easily seen that if b is an element of the set $\{b_j\}$, then $-b$ is also an element of the set. Hence, the set $\{b_j^2\}$ contains no more than $\frac{1}{2}(q^{2k} - 1)$ elements. Since the b_j are distinct, it follows from the remark immediately preceding Lemma 3 that the set $\{b_j^2\}$ contains exactly $\frac{1}{2}(q^{2k} - 1)$ elements. Finally, since there are $\frac{1}{2}(q^{2k} - 1)$ elements of M which are non-squares in M , it is seen that the set $\{b_j^2\}$ is precisely the set of all non-square elements of M . This completes the proof of the lemma.

THEOREM 2. *If A is an algebra satisfying the hypotheses of Lemma 4, then A has a basis $1, f, f^2, f^3$, with multiplication given by*

$$(1) \quad (f^2)^2 = \alpha_0 + \alpha_1f^2, \quad (f^3)^2 = \beta_0 + \beta_1f + \beta_2f^2 + \beta_3f^3, \\ f^2f^3 = \gamma_0 + \gamma_1f + \gamma_2f^2 + \gamma_3f^3, \quad f^3f^3 = \delta_0 + \delta_1f + \delta_2f^2 + \delta_3f^3.$$

Proof. First, note that $f^2f = ff^2$ by commutativity, and hence that f^3 is unambiguous. As in Lemma 4 let M be the subalgebra of A of order 2 over F . Then M has a basis $1, e$, with $e^2 = \phi$, where ϕ is a non-square in F . Let the

elements of M be ordered in some way and denote by $\eta_0 + \eta_1 e$ the first non-square in M with respect to this ordering. By Lemma 4 there exists an element $f \in A - M$ such that $f^2 = \eta_0 + \eta_1 e$. Furthermore, it is clear that $1, f, f^2, f^3$ are linearly independent over F . Define a_0 and a_1 by $a_0 = \phi\eta_1^2 - \eta_0^2$, $a_1 = 2\eta_0$, then clearly

$$\begin{aligned}(f^2)^2 &= (\eta_0 + \eta_1 e)^2 = \eta_0^2 + \phi\eta_1^2 + 2\eta_0\eta_1 e = \eta_0^2 + \phi\eta_1^2 + 2\eta_0 f^2 - 2\eta_0^2 \\ &= a_0 + a_1 f^2.\end{aligned}$$

Note that the constants a_0 and a_1 depend only upon the choice of $\phi \in F$ and the ordering of the elements of M . Nothing can be said about the twelve constants $\beta_i, \gamma_i, \delta_i$ ($i = 0, 1, 2, 3$).

The theorem of Dickson, mentioned earlier, states that an algebra A satisfying the hypotheses of Lemma 4 has a basis $1, f, f^2, f^3$ with multiplication given by $(f^2)^2 = \xi_0 + \xi_1 f + \xi_2 f^2 + \xi_3 f^3$, and $(f^3)^2, ff^2, f^2 f^2$ the same as in (1), where the polynomial $\lambda^4 - \xi_3 \lambda^3 - \xi_2 \lambda^2 - \xi_1 \lambda - \xi_0$ is irreducible in $F[\lambda]$.

4. Finite commutative division algebras of order 3. Let A be a commutative division algebra of order 3 over a field $F = GF(q^k)$, $q > 2$. It has been shown by Dickson [3; 4; 6; 7] that if A is not associative, then A has a basis $1, e, e^2$ with multiplication given by

$$(2) \quad ee^2 = \gamma + \delta e, \quad e^2 e^2 = -\delta^2 - 8\gamma e - 2\delta e^2,$$

where $\lambda^3 - \delta\lambda - \gamma$ is irreducible in $F[\lambda]$, and conversely if $\lambda^3 - \delta\lambda - \gamma$ is irreducible in $F[\lambda]$, then the algebra over F with basis $1, e, e^2$ and multiplication given by (2) is a division algebra. Dickson has shown further that there is at most one commutative, non-associative division algebra with unit element of order 3 over a Galois field $GF(q^k)$, $q > 2$, and that this unique non-associative division algebra has as its automorphism group relative to the base field, the cyclic group of order 3.

The question of the existence of finite, non-associative, commutative division algebras of order a prime $p > 3$ appears to be rather difficult to answer. In fact, to the best of the author's knowledge, there exist no examples of such algebras. The most obvious approach to the problem is a study of p -ary, p -ic forms, p a prime, over a Galois field, which vanish only when each of the p variables vanishes. The connection between these two problems is seen by noting that (i) an algebra A , of order p over a field F , is a division algebra if and only if $|R(x)| = 0$ implies that $x = 0$, where $R(x)$ is the linear transformation defined by $aR(x) = ax$ for all $a \in A$; and (ii) $|R(x)|$ is a p -ary, p -ic form³ in the p components of x . The problem of determining all "definite" p -ary, p -ic forms over a Galois field has been solved by Dickson [7] for the case $p = 3$, and this is one reason for a fairly complete knowledge of finite, commutative division algebras of order 3.

³For a further discussion of this see Bruck [1] and Dickson [4].

In the next section an important method due to Dickson [5, p. 515] will be employed to obtain non-associative, commutative division algebras of order $2n$ over a Galois field, whose automorphism group relative to the base field is the cyclic group of order $2n$. This result of Dickson may be summarized as follows: if

$$f(\lambda) = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \dots \pm a_n \in F[\lambda]$$

(where a_n is a non-square in the field F) is an irreducible, normal, cyclic polynomial, if ρ is a zero of $f(\lambda)$ and S a generating automorphism of the automorphism group of $F(\rho)$ relative to F , then the set A of all ordered pairs (x, y) , $x, y \in F(\rho)$ is a division algebra of order $2n$ over F under the operations

$$a(x, y) = (x, y)a = (ax, ay), \quad a \in F,$$

$$(x, y) + (a, b) = (x + a, y + b), \quad (x, y)(a, b) = (xa + ySbSp, ya + xb).$$

5. The existence of finite non-associative division algebras. As noted earlier, the non-zero elements of a division algebra with unit element form a loop under multiplication. It is interesting, and somewhat useful, as the next theorem will indicate, to be able to determine whether or not the set of non-zero elements of a given division algebra with unit contains a subloop of index 2. No non-associative, commutative, finite, division algebras with this property are known to the author. However, the set of all squares in a Galois field is a subgroup of index 2 in the group of non-zero elements, and it is easy to see that if the loop of non-zero elements of a finite, commutative division algebra contains a subloop of index 2, then this subloop is necessarily the set of all squares. It should be mentioned at this point that there exist non-associative division algebras, not finite, with this property. Thus, in the linearly ordered algebras constructed by Zelinsky [12] the set of all positive elements is the desired subloop of index 2.

The following theorem is closely related to the result of Dickson referred to at the end of the last section and will be applied to the Galois fields.

THEOREM 3. *Let A be a division algebra (not necessarily associative) with unit element of order n over a field K . Denote by G the set of all non-zero elements of A . If the loop G contains a subloop H of index 2 in G , then the set A^* of all ordered pairs (x, y) , $x, y \in A$, is a division algebra with unit element of order $2n$ over K under the operations*

$$(3) \quad a(x, y) = (x, y)a = (ax, ay), \quad a \in K,$$

$$(x, y) + (z, w) = (x + z, y + w), \quad (x, y)(z, w) = (xz + [yUwV]e, yz + xw),$$

where e is any fixed element of $G - H$, and U, V are non-singular linear transformations of A such that $HU = HV = H$.

Proof. First note that if $(x, y)(z, w) = (0, 0)$, $(x, y) \neq (0, 0)$, $(z, w) \neq (0, 0)$ then x, y, z, w are all different from zero. Suppose that there exist elements $x, y, z, w \in G$ such that $(x, y)(z, w) = (0, 0)$. Then by (3) it is seen that

$$(4) \quad \text{(i) } xz + (yUwV)e = 0, \quad \text{(ii) } yz + xw = 0.$$

Eliminating x from equations (4) it is found that

$$(5) \quad yz = yUR(wV)R(e)R(z)^{-1}R(w).$$

Now, denote by 1 and -1 the elements of C_2 , the cyclic group of order 2, and define the mapping F of G upon C_2 by $F(x) = 1$, if $x \in H$, and $F(x) = -1$, if $x \notin H$. Clearly F is a homomorphism of G upon C_2 , and it is easily verified that $F[aR(b)^{-1}] = F(a)F(b)$, and $F(aU) = F(aV) = F(a)$, for all $a, b \in G$. From equation (5) it follows that $F(yz) = F(y)F(w)F(e)F(z)F(w)$, which implies that $F(e) = 1$, contrary to the hypothesis $e \notin H$. Thus, $(x, y) (z, w) = (0, 0)$ implies either $x = y = 0$, or $z = w = 0$. The absence of divisors of zero in the set of non-zero elements of A^* insures, in this case, that they form a loop with respect to the multiplication defined in (3). It is readily verified that the remaining postulates for a division algebra are satisfied by A^* . This completes the proof of the theorem.

It is easily seen that if the algebra A of Theorem 3 is a Galois field, if $U = V = I$, and if e is a non-square in A , then the algebra A^* of Theorem 3 is simply $A(e)$, the quadratic extension of A . It should be noted, however, that if $U = V \neq I$, then the algebra A^* is not associative. Thus, let $A = GF(q^{nk})$, so that A is an associative division algebra of order n over $F = GF(q^k)$. Choose U and $V = U$ from the set of automorphisms of A relative to F . Then, if $e \notin H$, that is, if e is a non-square in A , the algebra A^* is a commutative, non-associative division algebra with unit element of order $2n$ over F . The following theorem shows that under certain conditions the automorphism group A^* contains the cyclic group of order $2n$.

THEOREM 4. *Let F be the Galois field $GF(q^k)$, $q > 2$, and A the field $GF(q^{nk})$, where n is any positive integer. Let S be a generating automorphism of the automorphism group of A relative to F . If A^* is the non-associative algebra of order $2n$ over F defined as in Theorem 3 with $U = V = S$, and e any non-square in A , then the automorphism S of A may be extended to an automorphism T of A^* relative to F . Furthermore, if n is odd, T has period $2n$.*

Proof. Since e is a non-square in A , it follows that e^{-1} and eS are non-squares in A , and hence that $e^{-1} \cdot eS$ is a square. Denote by c either of the two square roots of $e^{-1} \cdot eS$. For any $x \in A$, let $N(x)$ be the usual norm of x over F , that is $N(x) = x \cdot xS \cdot xS^2 \cdots xS^{n-1}$, then clearly $N(c) = \pm 1$. Note that if n is odd, c may be chosen so that $N(c) = -1$. Let $f = cS^{n-1}$, and define the linear transformation T of A^* by $(a, b)T = (aS, f \cdot bS)$. It is readily verified that T is an endomorphism of A^* and that $T^{2n} = I$. These two facts imply that T is an automorphism of A^* . When n is odd, f may be chosen so that $N(f) = -1$. If j is the period of T , and b a non-zero element of A , it is readily verified that $(a, b)T^j \neq (a, b)$. It follows that $j \neq n$. However, $(a, b)T^j = (aS^j, *) = (a, b)$, for all $a, b \in A$, whence $aS^j = a$, for all $a \in A$. Hence $j = hn$, for some positive integer h . Finally, $T^{2n} = I$ implies that $2n = rj$, for some positive integer r . From these relations involving the integers j, h, n , and r it may be inferred that $j = 2n$. This completes the proof of the theorem.

Before proceeding to the next theorem it is necessary to make the following definition. Let A be a division algebra with unit element of order n over a field F . If A has a basis $1, e, e^2, \dots, e^{n-1}$, with multiplication given by⁴

$$(6) \quad e^n = \phi, \quad e^i \cdot e^j = \phi(i, j)e^{i+j}, \quad i, j = 1, \dots, n-1,$$

where it is understood that $i+j$ is reduced modulo n , and $\phi, \phi(i, j) \in F$, then A is said to have a *cyclic basis* relative to F .

THEOREM 5. *Let n be any positive integer and $F = GF(q^k)$, $q > 2$, a Galois field with $q^k = 2ns + 1$, for some positive integer s . Then there exists a commutative, non-associative division algebra A^* of order $2n$ over F with the following properties:*

- (i) A^* has a cyclic basis relative to F ,
- (ii) the automorphism group of A^* relative to F is the cyclic group of order $2n$,
- (iii) A^* contains a unique associative subalgebra of order n over F isomorphic to the field $GF(q^{nk})$.

Proof. Note that $q^k = 2ns + 1$ is equivalent to the statement: F contains $2n$ distinct $(2n)$ th roots of unity. In this case there exists a polynomial $\lambda^n - \phi$, irreducible in $F[\lambda]$, and such that if n is even, $-\phi$ is a non-square in F , and if n is odd, ϕ is a non-square in F . Let $A = GF(q^{nk})$ and denote by S a generating automorphism of the automorphism group of A relative to F . Now, there exists an element $e \in A$, which satisfies $\lambda^n - \phi = 0$, and is a non-square in A . With this choice of e , and with $U = V = S$, the algebra A^* of Theorem 3 is clearly a commutative division algebra with unit element of order $2n$ over F . First it will be shown that A^* possesses a cyclic basis. Denote by $\zeta \in F$ a primitive $(2n)$ th root of unity. Then ζ^2 is a primitive n th root of unity and without loss of generality it may be assumed that $eS = \zeta^2 e$. If $g = (0, e^{n-1})$, it is easily verified, by finite induction on i , that the relations

$$(7) \quad (i) \quad g^{2i-1} = \alpha_{2i-1}(0, e^{n-i}), \quad (ii) \quad g^{2i} = \alpha_{2i}(e^{n-i}, 0),$$

where $\alpha_j \neq 0$, $\alpha_j \in F$ ($j = 1, \dots, 2n$) hold for $i = 1, \dots, n$. Since $1, e, e^2, \dots, e^{n-1}$ is a basis for A over F , it follows that $1, g, g^2, \dots, g^{2n-1}$ is a basis for A^* over F . Again, by induction it may be shown that there exist non-zero elements $\phi(r, s) \in F$ ($r, s = 1, \dots, 2n-1$) such that $g^r \cdot g^s = \phi(r, s)g^{r+s}$, where $r+s$ is reduced modulo $2n$ if necessary. Note also that, by relation (7, ii), $g^{2n} = \alpha_{2n}\phi \in F$. Thus, A^* has a cyclic basis relative to F .

Now, it is evident that the mapping T of A^* upon A^* defined by $g^i T = \zeta^{-1} g^i$ is an automorphism of period $2n$. Thus, the automorphism group of A^* relative to F contains the cyclic group of order $2n$. Let K be any automorphism of A^* relative to F and note that for the case $n = 2$, the uniqueness (see Theorem 1) of the subalgebra of order 2 implies that K induces an automorphism of this subalgebra. It will be assumed then that $n > 2$, and it will be shown that the

⁴The following convention is adopted for positive integral powers in a non-associative algebra: if x is any element of the algebra and i any positive integer, then x^i denotes the right power of x , defined by $x^i = x[R(x)]^{i-1}$.

subalgebra A is a unique associative subalgebra of A^* , of order n over F . Indeed, if it is assumed that K does not induce an automorphism of A , then K maps A into an isomorphic subalgebra B of A^* . Since $1, e, e^2, \dots, e^{n-1}$ is a basis for A over F , it follows that $(eK)^i$ ($i = 0, 1, \dots, n-1$) is a basis for B . Then, $eK = (a, b) \notin A$, so that $b \neq 0$. Let $f = eK = (a, b)$, then, since B is an associative subalgebra of A^* , it follows that $(f^2)^2 = f^2f$. This last equation, written in terms of a, b , is

$$[* , 4ab(a^2 + (bS)^2e)] = [* , 4a^2b + 2ab(bS)^2e + 2(aS)b(bS)^2e].$$

Equating the second "components" of $(f^2)^2$ and f^2f , it is found that $a = aS$. Thus $a \in F$, and denoting a by a , the relation $(f^2)^2 = f^2f$, in terms of a, b simplifies to

$$[a^4 + 6a^2(bS)^2e + (bS)^4e^2, *] = [a^4 + 6a^2(bS)^2e + (bS)^2(bS^2)^2eeS, *].$$

Equating the first "components" and noting that $eS = \zeta^2e$, it is seen that $(bS)^2 = \zeta^2(bS^2)^2$. This last equation may be written $(b^2S)S = \zeta^{-2}(b^2S)$, from which it follows that $b^2S = \psi e^{n-1}$, $\psi \in F$. Now,

$$f^2 - 2af + a^2 - \psi\phi = (f - a)^2 - \psi\phi = (0, b)^2 - \psi\phi = 0.$$

Thus, $1, f, f^2$ are linearly dependent over F , which implies $n = 2$, contrary to the assumption $n > 2$. This shows that the subalgebra A is the only associative subalgebra of order n over F , contained in A^* . In particular then, the arbitrary automorphism K of A^* induces an automorphism of A , so that $(e, 0)K = \zeta^{2j}(e, 0)$ for some positive integer j . Since $[(0, 1)]^2 = (e, 0)$, it follows that $(0, 1)K = \pm \zeta^j(0, 1)$; hence

$$gK = (0, e^{n-1})K = [(0, 1)] [(e^{n-1}, 0)K] = \pm \zeta^{j(2n-1)}g.$$

Thus, gK is the product of g and a $(2n)$ th root of unity, that is, K coincides with a power, T^r , of the automorphism T defined above. It is seen then that the automorphism group of A^* relative to F is the cyclic group of order $2n$. This completes the proof of the theorem.

6. Finite division algebras of order n whose automorphism group contains the cyclic group of order n . Let A be a division algebra of order n over an arbitrary field F . If A has a cyclic basis relative to F and if m is a divisor of n , then it is evident that A contains a subalgebra of order m . The following theorem gives a sufficient condition, not quite as immediate as the above, for a finite division algebra of order n to contain a subalgebra of order m , where m is any divisor of n .

THEOREM 6. Let A be a division algebra of order n over a field $F = GF(q^t)$, where $n = hq^t$, $(h, q) = 1$. Let T be an automorphism of A relative to F with period n . If the minimum function of T is of degree n , then, for every divisor m of n , A contains a subalgebra A_m , of order m over F , whose automorphism group relative to F contains the cyclic group of order m .

Proof. Let Ω be the root field of the polynomial $\lambda^h - 1 \in F[\lambda]$. Then there exists an element $\zeta \in \Omega$ such that

$$\lambda^h - 1 = \prod_{i=0}^{h-1} (\lambda - \zeta^i)$$

in $\Omega[\lambda]$. Since T satisfies $\lambda^h - 1 = 0$, and since its minimum function is of degree n , it follows that $\lambda^n - 1 = 0$ is the minimum equation of T . Now in $\Omega[\lambda]$,

$$\lambda^n - 1 = (\lambda^h - 1)^{e'} = \prod_{i=0}^{h-1} (\lambda - \zeta^i)^{e'}.$$

Thus, there exists a basis for the algebra A_0 such that a representation for T is given by [9, p. 128, Theorem 65]

$$T = E_0 \oplus E_1 \oplus \dots \oplus E_{h-1}, \quad E = \zeta^i I + E \quad (i = 0, 1, \dots, h-1),$$

where I is the $q^i \times q^i$ identity matrix and E is the $q^i \times q^i$ matrix with 1 everywhere in the diagonal just below the main diagonal and zeros elsewhere. If m is a divisor of n , let q^r be the highest power of q which divides m , so that $m = sq^r$ (s, q) = 1, and s divides h . Since $(s, q) = 1$, it is seen that the field Ω contains s distinct s th roots of unity, each of which is also an h th root of unity. Thus in the matrix T^s given by $T^s = E_0^s \oplus E_1^s \oplus \dots \oplus E_{h-1}^s$, exactly s of the components E_i^s will have 1 everywhere in the main diagonal and the remaining $h - s$ components will have diagonal elements different from 1. Let E_j^s be one of the components with 1 along the main diagonal, that is, let ζ^j be one of the s th roots of unity. Then,

$$E_j^s = (\zeta^j I + E)^s = I + s\zeta^{j(s-1)}E + \binom{s}{2}\zeta^{j(s-2)}E^2 + \dots + E^s = I + EF,$$

where

$$F = s\zeta^{j(s-1)}I + \binom{s}{2}\zeta^{j(s-2)}E + \dots + E^{s-1}.$$

By the definition of E , and since $(s, q) = 1$, F is non-singular. Furthermore,

$$E_j^m - I = (E_j^s - I)^{q^r} = E^{q^r} \cdot F^{q^r},$$

and hence the nullity of $E_j^m - I$ is equal to the nullity of E^{q^r} . Noting that the first v rows of E^s consist entirely of zeros, and that the remaining $q^i - v$ rows are linearly independent, it follows that the nullity of the matrix $E_j^m - I$ is exactly q^r . If ζ^p is one of the $h - s$ h th roots of unity which is not an s th root of unity, then clearly $E_p^m - I$ has nullity zero. Thus, in the expression of $T^m - I$ as a direct sum, given by⁵

$$T^m - I = (E_0^m - I) \oplus (E_1^m - I) \oplus \dots \oplus (E_{h-1}^m - I),$$

exactly s of the components have nullity q^r , and the remaining components have nullity zero. Thus, the nullity of $T^m - I$ is $sq^r = m$. Since the nullity of a

⁵In $T^m - I$ it is understood that I denotes the $n \times n$ identity matrix.

matrix with elements in a field F is invariant under an extension of F , it follows that in the original algebra A over the given field F there exist exactly m linearly independent elements in the null space of $T^m - I$. Thus, the subalgebra A_m , consisting of all elements of A which are mapped into themselves by T^m , is of order m over F . It is obvious that the automorphism T of A induces an automorphism T' of A_m of period m . Hence the group of automorphisms of the algebra A_m relative to F contains the cyclic group of order m . This completes the proof.

Suppose that F is a Galois field, $F = GF(q^k)$, and $n = hq^t$, $(h, q) = 1$. If h divides $q^k - 1$, then F contains h distinct h th roots of unity. If A is an algebra of order n over F whose automorphism group relative to F contains the cyclic group of order n , then it is seen that every h th root of unity is a characteristic root of any automorphism T of A which generates the cyclic group of automorphisms of order n . Thus, if

$$M = T^t,$$

so that the transformation M has period h , then it is clear that every characteristic root of M is an h th root of unity. Furthermore, the set of distinct characteristic roots of M is a subgroup of the set of h th roots of unity, for if η_1, η_2 , are any two characteristic roots, then there exist $a_1, a_2, \in A, a_1 \neq 0, a_2 \neq 0$, such that $a_1 M = \eta_1 a_1, a_2 M = \eta_2 a_2$, whence

$$(a_1 a_2) (M - \eta_1 \eta_2 I) = 0, \quad a_1 a_2 \neq 0,$$

so that $\eta_1 \eta_2$ is a characteristic root. If the set of distinct characteristic roots does not contain all of the h th roots of unity, then it is the set of s th roots of unity for some $s < h$, with s dividing h . Since a basis for A may be chosen in such a way that M is represented by a diagonal matrix, whose diagonal elements are s th roots of unity, it follows that $M^s = I$, contrary to the hypothesis that M is of period h . Thus, every h th root of unity is a characteristic root of M . Finally, since the mapping

$$\alpha \rightarrow \alpha^{q^t}$$

of F upon F , is a permutation of the h th roots of unity, it is seen that every characteristic root of M is a characteristic root of T . If $t > 0$, it is not known whether or not the minimum function of T is of degree n . However, if $t = 0$, so that $n = h \mid (q^k - 1)$, then the previous remarks indicate that the minimum function of T is of degree n , and Theorem 6 is applicable. In fact, in this case, it is possible to choose a basis $1, e_1, \dots, e_{n-1}$ for A over F in such a way that $e_i T = \zeta^i e_i$, where ζ is a primitive n th root of unity. It is readily verified that the multiplication for A is given by (6). Thus the following theorem has been proved.

THEOREM 7. *If A is a division algebra of order n over a field $F = GF(q^k)$, where $q^k = hn + 1$ for some positive integer h , and if the automorphism group of A*

relative to F contains the cyclic group of order n , then A has a cyclic basis relative to F .

Added in proof. The question discussed in the second paragraph of §4 has been answered by A. A. Albert. See abstract 421, Bull. Amer. Math. Soc., vol. 57 (1951), p. 457.

REFERENCES

1. R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc., vol. 56 (1944), 141-198.
2. ———, *Contributions to the theory of loops*, Trans. Amer. Math. Soc., vol. 60 (1946), 245-354.
3. L. E. Dickson, *On finite algebras*, Nachr. Ges. Wiss. Göttingen (1905), 358-393.
4. ———, *Linear algebras in which division is always uniquely possible*, Trans. Amer. Math. Soc., vol. 7 (1906), 370-390.
5. ———, *On commutative linear algebras in which division is always uniquely possible*, Trans. Amer. Math. Soc., vol. 7 (1906), 514-522.
6. ———, *On linear algebras*, Amer. Math. Monthly, vol. 13 (1906), 201-205.
7. ———, *On triple algebras and ternary cubic forms*, Bull. Amer. Math. Soc., vol. 14 (1907-08) 160-169.
8. Harriet Griffin, *The abelian quasigroup*, Amer. J. Math., vol. 62 (1940), 725-734.
9. C. C. MacDuffee, *Vectors and matrices* (Carus Mathematical Monographs, No. 7, 1943).
10. E. H. Moore, *A double infinite system of simple groups*, International Mathematical Congress, 1893.
11. J. H. M. Wedderburn, *A theorem on finite algebras*, Trans. Amer. Math. Soc., vol. 6 (1905), 340-352.
12. Daniel Zelinsky, *Nonassociative valuations*, Bull. Amer. Math. Soc., vol. 54 (1948), 175-183.

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ON D. E. LITTLEWOOD'S ALGEBRA OF S-FUNCTIONS

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1. Introduction. Several papers have been written on the "new" multiplication of S-functions since Littlewood [3, p. 206] first suggested the problem. M. Zia-ud-Din [13] calculated the case $\{m\} \otimes \{n\}$ for $mn \leq 12$, making use of the tables of the characters of the symmetric group of degree mn . Later Thrall [10, pp. 378-382] developed explicit formulae for the cases $\{m\} \otimes \{2\}$, $\{m\} \otimes \{3\}$, $\{2\} \otimes \{m\}$ (where m is any integer). Recently Todd [12] has obtained a formula for the factors $\{\mu\} \otimes S_n$ as sums of irreducible characters $\{\sigma\}$. This reduces the problem of calculating $\{\mu\} \otimes \{\lambda\}$ to the ordinary multiplication of S-functions [3, p. 94]. General solutions to the problem have also been obtained by Thrall [10; p. 375] and by Robinson [7; 8]. For these general results, however, the actual calculations are quite laborious in most cases.

In this paper a method of computing the general case $\{m\} \otimes \{4\}$ is developed and a formula is obtained (independently of Todd's method) for expressing the factors $l_{n(m)}$ as sums of S-functions $\{\sigma\}$. This formula provides a very brief method of calculating $l_{n(m)}$ and is easily adapted to recursive computation. The method of calculating $\{m\} \otimes \{4\}$ is also extended to cover all the remaining partitions of four. This method has been applied to calculate the products $\{7\} \otimes \{4\}$, $\{7\} \otimes \{2,1^3\}$ in full.

2. Preliminary definitions and lemmas. Using Thrall's notation [10, p. 374], $l_{n(m)} = \{m\} \otimes S_n$,

$$(1) \quad \{m\} \otimes \{\mu\} = \sum_{\sigma} \frac{\chi^{(\mu)}(\mu)}{\beta_1! \dots \beta_r!} \binom{l_{1(m)}}{1}^{\beta_1} \dots \binom{l_{r(m)}}{r}^{\beta_r}.$$

Hence, if the $l_{n(m)}$ are known as sums of S-functions the product $\{m\} \otimes \{\mu\}$ may be computed by the ordinary multiplication of S-functions.

In proving the direct and recursion formulae for $l_{n(m)}$ we will make use of the following three lemmas.

Definitions of Young diagram, n -hook, removal of an n -hook, star diagram and δ -number are given in [9].

LEMMA 1. *Let $(\sigma) = (\sigma_1, \dots, \sigma_n)$ be a partition of mn into n or less parts, and suppose that the numbers $r_1 = \sigma_1 + n - 1$, $r_2 = \sigma_2 + n - 2$, \dots , $r_n = \sigma_n$ are all incongruent (mod n). Then the necessary and sufficient condition on s and k that a hook of length ns with top right node lying in the k th row may be removed from $[\sigma]$ is that $r_k = \sigma_k + (n - k) \geq ns$.*

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*Proof.*¹ Since r_1, \dots, r_n are incongruent (mod n), the classes of congruent δ -numbers are exactly r_1, \dots, r_n and the corresponding diagrams are simply vertical lines of r_1, \dots, r_n nodes respectively. Hence for an ns -hook to be removable (with foot in the k th column), it is necessary and sufficient that $r_k \geq ns$ [9, p. 85, Theorem C].

LEMMA 2. Let (σ) be a partition of mn into n or less parts; then $[\sigma]$ has no n -core² if and only if the numbers r_1, \dots, r_n are incongruent (mod n).

Proof. This lemma follows at once from the criterion (7.12) given in [11, p. 722].

LEMMA 3. Let (σ) be a partition of mn defined as in Lemma 1. Then a hook of length kn may be added to $[\sigma]$ commencing with the lower left node at the end of any of the n rows of $[\sigma]$ (including cases where $\sigma_i = 0$) and a diagram $[\sigma']$ associated with a partition of $n(m+k)$ into n or less parts will result.

Proof. Let the top right node of the annexed hook lie in the $(p+1)$ th row of $[\sigma']$. Then if $[\sigma']$ is not a right diagram we have

$$(\sigma_p - \sigma_{p+1} + 1) + \dots + (\sigma_{p+s-1} - \sigma_{p+s} + 1) = nk$$

for some value of s . That is, $\sigma_p = \sigma_{p+s} - s$ (mod n), which contradicts the assumption that the r_i are incongruent.

3. **Formulae for $t_{n(m)}$.** The following theorems give direct and recursion formulae for $t_{n(m)}$.

THEOREM 1. $t_{n(m)} = \sum \phi_\sigma \{\sigma\}$ where (σ) ranges over all partitions of nm ; ϕ_σ is zero if (σ) has more than n parts or if the Young diagram $[\sigma]$ associated with (σ) has a (non-zero) n -core. Otherwise $\phi_\sigma = \theta_\sigma$ where θ_σ is plus or minus one according as the sum of the leg-lengths of the removed n -hooks is even or odd.

*Proof.*³ Let $T_{n(m)} = \sum \phi_\sigma \{\sigma\}$ where $\sum \phi_\sigma \{\sigma\}$ satisfies all the conditions stated in the theorem. We take as an induction hypothesis that $T_{n(h)} = t_{n(h)}$ for all $h < m$. Now we have [10, p. 374]

$$t_{n(1)} = S_n = \{n\} - \{n-1, 1\} + \{n-2, 1^2\} + \dots + (-1)^{n-1} \{1^n\} = T_{n(1)}.$$

Hence the theorem is true for $m = 1$. Now in general,

$$t_{n(m)} = \sum_{\beta} \frac{1}{\beta_1! \dots \beta_m!} \left(\frac{S_n}{1} \right)^{\beta_1} \dots \left(\frac{S_{nm}}{m} \right)^{\beta_m}.$$

We will show (i) $T_{n(m)}$, $t_{n(m)}$ have the same derivatives with respect to S_{nk} ($k = 1, \dots, m$) and (ii) $T_{n(m)}$ is a function of S_{nk} ($k = 1, \dots, m$) only.

¹I am indebted to Professor R. A. Staal for this proof which is shorter than my original one.

²When all possible n -hooks have been removed from a diagram the resulting diagram is called its n -core. The n -core and θ_σ are independent of the order of removal of the hooks [5], [6].

³For a method of evaluating the θ_σ which does not lead to the recursion formula see [1].

Now [4, p. 107]

$$(2) \quad k \frac{\partial t_{n(m)}}{\partial S_{nk}} = t_{n(m-k)} = \sum_{\mu} \phi_{\mu} \{\mu\},$$

where (μ) ranges over all partitions of $n(m-k)$ and the ϕ_{μ} are, by the induction hypothesis, as described in the theorem. Also [4, p. 133]

$$(3) \quad (nk) \frac{\partial T_{n(m)}}{\partial S_{nk}} = \sum_{\sigma} \phi_{\sigma} \sum_{j=1}^n \{\sigma_1, \dots, \sigma_j - nk, \dots, \sigma_n\},$$

or in the language of hooks:

$$(4) \quad (nk) \frac{\partial T_{n(m)}}{\partial S_{nk}} = \sum_{\sigma} \phi_{\sigma} \sum_i (-1)^{h_i} \{\sigma^i\},$$

where $\{\sigma^i\}$ is obtained from $\{\sigma\}$ by removing an nk -hook of leg-length h_i commencing in the i th row and the summation is over all values of i (rows) from which such a hook may be removed. Now multiplying (2) by n we have:

$$(5) \quad (nk) \frac{\partial t_{n(m)}}{\partial S_{nk}} = n \sum_{\mu} \phi_{\mu} \{\mu\}.$$

Hence we must show the right sides of (4), (5) to be equal. For fixed (μ) we label the n values of $\phi_{\mu} \{\mu\}$ occurring on the right side of (5) by $\phi_{\mu} \{\mu\}_1, \dots, \phi_{\mu} \{\mu\}_n$. Consider $\phi_{\mu} \{\mu\}_r$ (for $\phi_{\mu} \neq 0$), by Lemma 3 an nk -hook may be added to $[\mu]$ starting (bottom left node) at the r th row and a new diagram will result. Let this annexed hook terminate in the j th row, then denoting the augmented diagram by $[\sigma]$ we must show $\phi_{\mu} \{\mu\}_r = \phi_{\sigma} (-1)^{h_i} \{\sigma_j\}$ where $h_j = r - j$, that is, we must show $\phi_{\mu} = \theta_{\sigma} (-1)^{h_i}$. Now by Lemma 1 the nk -hook which is deleted from $[\sigma]$ to yield $[\mu]$ may be partitioned into k n -hooks which may be removed in order, starting at the top right node of the nk -hook. Again by Lemma 1, each deletion leaves a new diagram; hence the bottom left node of a given n -hook must lie in the same row as the top right node of its successor. Let the i th removed n -hook terminate in the q_i th row and commence in the q_{i-1} th row; then $q_0 = j$ and $q_k = r$. Now the sum of the leg-lengths of these removed hooks is

$$(q_1 - q_0) + \dots + (q_k - q_{k-1}) = q_k - q_0 = r - j,$$

which is the leg-length (h_j) of the nk -hook. But by the induction assumption

$$\phi_{\mu} = \theta_{\sigma} (-1)^{(q_1 - q_0) + \dots + (q_k - q_{k-1})} = \theta_{\sigma} (-1)^{h_i},$$

as required. Similarly there corresponds to each

$$\theta_{\sigma} (-1)^{h_i} \{\sigma^i\}$$

a unique $\phi_{\mu} \{\mu\}_r$. To demonstrate (ii) we write [3, p. 86; 10, p. 374]

$$(6) \quad T_{n(m)} = \sum_{\sigma} \phi_{\sigma} \{\sigma\} = \sum_{\sigma} \phi_{\sigma} \sum_p \frac{h_p}{(mn)!} \cdot \chi_{\sigma}^p S_p$$

where

$$S_p = (s_1)^{p_1} \dots (s_{nm})^{p_{nm}}.$$

Now the coefficient of S_ρ on the right of (6) is

$$\frac{h_\rho}{(mn)!} \sum_{\sigma} \phi_\sigma \chi_\sigma^\sigma.$$

Now ϕ_σ has been shown [11] to be expressible as

$$\phi_\sigma = \sum_{\alpha} c_{\alpha} \chi_{\alpha}^{\sigma}$$

where α ranges over partitions of mn of the form $(\beta)_n$ where (β) is a partition of m and $(\beta)_n$ is the partition of mn obtained from (β) on multiplying each element of (β) by n , that is, $(\beta)_n = (\beta_1 n, \dots, \beta_m n)$. Hence from the orthogonality relations for the characters of the symmetric group the coefficient of

$$S_\rho = \frac{h_\rho}{(mn)!} \sum_{\sigma} \sum_{\alpha} c_{\alpha} \chi_{\alpha}^{\sigma} \chi_{\rho}^{\sigma}$$

is zero if (ρ) is not of the form $(\beta)_n$ also. Hence $T_{n(m)}$ is a function of the $S_{\alpha k}$ ($k = 1, \dots, m$) only.

THEOREM 2. Let $t_{n(m)} = \sum \phi_{\sigma} \{\sigma\}$, then $t_{n(m+1)}$ is obtained recursively as follows. To each $\{\sigma\}$ associated with a partition (σ) for which ϕ_{σ} is not zero we add an n -hook in all possible ways whose top right node lies in the first row of the augmented diagram $\{\sigma'\}$. Then $t_{n(m+1)} = \sum \phi_{\sigma'} \{\sigma'\}$ where $\phi_{\sigma'} = \phi_{\sigma} (-1)^k$ where k is the leg-length of the annexed hook.

Proof. The proof follows at once from Lemmas 1, 2, 3 and Theorem 1.

In recent papers [7, 8, 12], Robinson and Todd have given independent methods for evaluating $\{\mu\} \otimes \{\lambda\}$ by step by step building processes. Robinson gives a systematic procedure (in place of Littlewood's more or less empirical methods) by means of which the irreducible components of $\{\mu\} \otimes \{\lambda\}$ can be determined. In this general method the recursion is from n to $n+1$. Todd gives a general method and also treats the restricted case $\{m\} \otimes S_n = t_{n(m)}$ studied here. He gives recursion formulae by means of which $t_{n(m)}$ may be determined if $t_{n-1(m)}$ and $t_{n(m-1)}$ are both known. In the above methods the quantity θ_r is made use of throughout.

4. The product $\{m\} \otimes \{4\}$. We now develop a method for computing the general case $\{m\} \otimes \{4\}$. From the calculations for $\{m\} \otimes \{4\}$ (for a specific value of m) $\{m\} \otimes \{2, 1^2\}$ is obtained by inspection. A modification of this method is also given for computing $\{m\} \otimes \{3, 1\}$ and $\{m\} \otimes \{1^4\}$. The remaining case, $\{m\} \otimes \{2^2\}$, follows immediately from the calculations for $\{m\} \otimes \{4\}$ and $\{m\} \otimes \{1^4\}$; hence the method applies to every partition of four.

Writing t_i for $t_{i(m)}$ we have, from (1) of §2;

$$\{m\} \otimes \{4\} = \frac{1}{24} (t_1^4 + 6t_1^2 t_2 + 3t_2^3 + 8t_2 t_1 + 6t_4).$$

Rearranging terms we have

$$(7) \quad \{m\} \otimes \{4\} = \frac{1}{12} \left[\frac{3}{2} (t_1^2 + t_2)^2 - t_1^4 + 4t_3t_1 + 3t_4 \right].$$

Now [10, p. 380]

$$\frac{1}{2} (t_1^2 + t_2) = \{m\} \otimes \{2\} = \sum_v \{2m - 2v, 2v\}, \quad v < \frac{1}{2}m.$$

It remains to develop explicit formulae for t_1^4 , t_3t_1 as sums of irreducible characters $\{\sigma\}$; t_4 being known by Theorem 1.

The following congruence relations will be used in the proofs which follow. Let

$$t_3t_1 = \sum_{\lambda} \theta_{\lambda}^{t_3t_1} \{\lambda\}, \quad t_4 = \sum_{\lambda} \theta_{\lambda}^{t_4} \{\lambda\}, \quad t_1^4 = \sum_{\lambda} \theta_{\lambda}^{t_1^4} \{\lambda\}.$$

Now

$$\theta_{\lambda}^{t_3t_1}, \quad \theta_{\lambda}^{t_4}, \quad \theta_{\lambda}^{t_1^4}$$

are integers or zero, hence we have, from (7),

$$(8) \quad \theta_{\lambda}^{t_3t_1} = \theta_{\lambda}^{t_1^4} \pmod{3},$$

$$(9) \quad \theta_{\lambda}^{t_4} = \theta_{\lambda}^{t_1^4} \pmod{2}.$$

We now derive a formula for $t_1^4 = \{m\}^4$. Let (λ) be an arbitrary partition $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $4m$ into four or less parts. We proceed to calculate the coefficient of $\{\lambda\}$ in $\{m\}^4$. To illustrate a term in the product $\{m\}^4$ diagrammatically we denote the Young diagrams $[m]_i$ by the respective notations:

$$x x x \dots x, o o o \dots o, * * * \dots *, - - - \dots -.$$

Then a diagram $[\lambda]$, corresponding to $\{\lambda\}$ must appear in the product $[m]_1 [m]_2 [m]_3 [m]_4$ as follows:

$$[\lambda] = \begin{array}{ccccccc} x & x & x & \dots & x & o & o & o & \dots & o & * & * & * & \dots & * & - & - & - & \dots & - \\ & & & & & o & o & o & \dots & o & * & * & * & \dots & * & - & - & - & \dots & - \\ & & & & & & & & & & * & * & * & \dots & * & - & - & - & \dots & - \\ & & & & & & & & & & & & & & & - & - & - & \dots & - \end{array}$$

Now labelling the set of nodes in the first row which arises from $[m]_1$ by u_{11} , in the second row by u_{12} etc., we have:

$$\begin{array}{ll} m = u_{11} & \text{and } \lambda_1 = u_{11} + u_{21} + u_{31} + u_{41} \\ m = u_{21} + u_{22} & \lambda_2 = u_{22} + u_{32} + u_{42} \\ m = u_{31} + u_{32} + u_{33} & \lambda_3 = u_{33} + u_{43} \\ m = u_{41} + u_{42} + u_{43} + u_{44} & \lambda_4 = u_{44} \end{array}$$

By a repeated application of the rule for the ordinary multiplication of S-functions we see that the necessary and sufficient conditions that a set of

integers u_{ij} form a Young diagram appearing in the product $\{m\}^4$ are the following:

- | | |
|--|---|
| (a) $\sum_i u_{ij} = \lambda_j$ | (e) $\lambda_2 \leq u_{11} + u_{21} + u_{31}$ |
| (b) $\sum_j u_{ij} = m$ | (f) $u_{33} \leq u_{22}$ |
| (c) $u_{ij} \geq 0$ | (g) $\lambda_3 \leq u_{22} + u_{32}$ |
| (d) $u_{22} + u_{32} \leq u_{11} + u_{21}$ | (h) $\lambda_4 \leq u_{33}$ |

Conditions (a), (b), (c) follow from the geometry of the Young diagram; (d), . . . , (h) follow from the rule for multiplying S-functions of type $\{m\}$. Now it follows from conditions (a), (b) that the quantities u_{33}, u_{22}, u_{23} determine all the u_{ij} uniquely. Relabelling these quantities i, j, k respectively, we write all the u_{ij} in terms of the quantities $i, j, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4, m$:

$$\begin{array}{ll} u_{11} = m & u_{31} = m - (i + j) \\ u_{21} = m - k & u_{44} = \lambda_4 \\ u_{22} = k & u_{43} = \lambda_3 - i \\ u_{32} = j & u_{42} = \lambda_2 - (k + j) \\ u_{33} = i & u_{41} = \lambda_1 + (i + j + k) - 3m \end{array}$$

Now rewriting (d), . . . , (h) we have:

$$\begin{array}{ll} (d') \quad k \leq \frac{1}{2}(2m - j) & (g') \quad \lambda_3 - j \leq k \\ (e') \quad k \leq 3m - \lambda_2 - (i + j) & (h') \quad \lambda_4 \leq i \\ (f') \quad i \leq k & \end{array}$$

Combining these inequalities with $u_{ij} \geq 0$, we obtain the following limits for i, j, k :

$$\begin{array}{l} \max(\lambda_4, \lambda_3 + \lambda_4 - m) \leq i \leq \min(m, \lambda_3) \\ 0 \leq j \leq \min(m - i, \lambda_2 - i) \\ N \leq k \leq M \end{array}$$

where
$$\begin{aligned} M &= \min(\frac{1}{2}(2m - j), \lambda_2 - j, 3m - (\lambda_2 + i + j)) \\ N &= \max(i, \lambda_3 - j, 3m - (\lambda_1 + i + j)) \end{aligned}$$

Now setting $K_{ij} = \max(0, 1 + M - N)$ we have

THEOREM 3.

$$t_1^4 = \{m\}^4 = \sum_{\lambda} \theta_{\lambda}^{t_1^4} \{\lambda\} \text{ where } \theta_{\lambda}^{t_1^4} = \sum_{i,j} K_{ij}$$

and i, j range over the values indicated above.

This formula illustrates the fact (which is easily proved directly in the general case) that if

$$\begin{aligned} \{m\}^n &= \sum_{\lambda} \theta_{\lambda}^{t_1^n} \{\lambda\}, \\ \{m+k\}^n &= \sum_{\lambda} \theta_{\lambda}^{t_1^n} \{\bar{\lambda}\} \end{aligned}$$

then if $(\bar{\sigma})$ is a partition of $n(m+k)$ with $\bar{\sigma}_n \geq k$, and if $(\sigma) = (\bar{\sigma}_1 - k, \dots, \bar{\sigma}_n - k)$, then

$$\theta_{\sigma}^{t_1^n} = \theta_{\bar{\sigma}}^{t_1^n}.$$

Hence we have a recursion formula for $\{m+1\}^n$ in terms of $\{m\}^n$ for all partitions (σ) of $(m+1)(n)$ with $\sigma_n \geq 1$.

The following theorem enables us to compute the quantity t_{31} by inspection.

THEOREM 4.

$$t_{31} = \sum_{\lambda} \theta_{\lambda}^{t_1 t_1} \{\lambda\}$$

where $\theta_{\lambda}^{t_1 t_1} = 1, 0, -1$ according as $\theta_{\lambda} t_1^4$ is congruent to 1, 0, -1 respectively (mod 3).

Proof. The congruence (mod 3) has been established (8). It remains to be shown that $\theta_{\lambda}^{t_1 t_1}$ is always 1, 0, or -1 . To show this we let $[10, p. 381]$ $t_3 = \sum g(\lambda') \{\lambda'\}$ where $g(\lambda')$ is 1, 0, -1 according as $(1 + \lambda_1 - \lambda_2)$ is congruent to 1, 0, -1 (mod 3), and (λ') ranges over all partitions of $3m$ into three or fewer parts. Now $t_1 = \{m\}$, hence

$$(10) \quad t_{31} = \sum_{\lambda'} g(\lambda') \{\lambda'\} \{m\} = \sum_{\lambda} \theta_{\lambda}^{t_1 t_1} \{\lambda\}.$$

Consider a partition (λ) of $4m$, then

$$\theta_{\lambda}^{t_1 t_1} = \sum_{\lambda'} g(\lambda'),$$

where the summation is over all partitions (λ') from which (λ) can be obtained on multiplying $\{\lambda'\}$ by $\{m\}$. Now consider which diagrams $\{\lambda'\}$ are obtained from $\{\lambda\}$ on deleting m nodes as indicated in (10) above. Since (λ') is a partition of $3m$ into three or fewer parts, this amounts to deleting $(m - \lambda_4)$ nodes from the first three rows of $\{\lambda\}$ in accordance with the rule for multiplying $\{\lambda'\} \{m\}$. Four cases arise:

- (i) $\lambda_1 - \lambda_2 \geq m - \lambda_4, \quad \lambda_2 - \lambda_3 \geq m - \lambda_4$
- (ii) $\lambda_1 - \lambda_2 \geq m - \lambda_4, \quad \lambda_2 - \lambda_3 < m - \lambda_4$
- (iii) $\lambda_1 - \lambda_2 < m - \lambda_4, \quad \lambda_2 - \lambda_3 \geq m - \lambda_4$
- (iv) $\lambda_1 - \lambda_2 < m - \lambda_4, \quad \lambda_2 - \lambda_3 < m - \lambda_4$

We will consider (i) in detail. The number of nodes which may be deleted from λ_3 is 0, 1, 2, \dots , $\min(\lambda_3 - \lambda_4, m - \lambda_4) = s$. We first delete zero nodes from λ_3 , $m - \lambda_4 - r$ from λ_1 and r from λ_2 ($r = 0, 1, \dots, m - \lambda_4$). This gives rise to the set of values for $g(\lambda')$ whose sum is indicated as T_1 below. We next delete one node from λ_3 , $m - \lambda_4 - (r + 1)$ from λ_1 and r from λ_2 ($r = 0, 1, \dots, m - \lambda_4 - 1$), giving rise to a set of values for $g(\lambda')$ with sum indicated as T_2 below. This process is continued to the $s = \min(\lambda_3 - \lambda_4, m - \lambda_4)$ step. Denoting the set of values 0, $-1, 1$, by x_1, x_2, x_3 , not necessarily respectively but in the same cyclic order, the sums T_1, \dots, T_s must then appear as follows:

$$\begin{aligned} T_1 &= x_1 + x_2 + x_3 + x_1 + x_2 + x_3 + x_1 + \dots + x_1 && (m - \lambda_4 + 1) \text{ terms} \\ T_2 &= x_3 + x_1 + x_2 + x_3 + x_1 + x_2 + \dots + x_j && (m - \lambda_4) \text{ terms} \\ T_3 &= x_2 + x_3 + x_1 + x_2 + x_3 + \dots + x_3 && (m - \lambda_4 - 1) \text{ terms} \end{aligned}$$

\dots

Now since $x_1 + x_2 + x_3 = 0$, we have at once:

$$\theta_{\lambda}^{t_1 t_2} = \sum_{\lambda'} g(\lambda') = T_1 + \dots + T_s = 0$$

if $s \equiv 0 \pmod{3}$, since each set of three rows has total sum zero. If $s \equiv 1 \pmod{3}$ the sum is simply T_s , which is obviously 1, -1 or zero in all cases. If $s \equiv 2 \pmod{3}$ we partition the final two rows T_{s-1}, T_s , as indicated below:

$$\begin{aligned} T_{s-1} &= \overbrace{x_1 + x_2} + x_3 \mid \overbrace{x_1 + x_2} + x_3 \mid x_3 + \dots + x_t, \\ T_s &= \quad \quad \quad \overbrace{x_3} + x_1 + x_2 \mid \overbrace{x_3} + x_1 + \dots + x_p. \end{aligned}$$

Hence $T_1 + \dots + T_s = T_{s-1} + T_s$ is 0, x_1 , or $x_1 + x_3$ which is obviously one of 0, 1, or -1 for all values of x_1, x_3 . Thus the proof for case (i) is complete. The cases (ii), (iii), (iv) give rise to a similar type of array of values as displayed above for case (i); again by direct calculation the total sum is seen to be 0, 1, or -1. This completes the proof of the theorem.

The remaining term of (7),

$$\frac{1}{4}(t_1^2 + t_2)^2 = [\sum \{2m - 2v, 2v\}]^2,$$

is computed directly by the ordinary multiplication of S-functions. This calculation is somewhat lengthy although it is a considerable simplification of the direct calculation of t_2^2 and $t_1^2 t_2$ independently.

5. The remaining partitions of four. We first consider the case $\{m\} \otimes \{2, 1^2\}$:

$$\begin{aligned} \{m\} \otimes \{2, 1^2\} &= \frac{1}{24}(3t_1^4 - 6t_1^2 t_2 - 3t_2^2 + 6t_4) \\ &= \frac{1}{12} \left[-\frac{3}{2}(t_1^2 + t_2)^2 + 3t_1^4 + 3t_4 \right]. \end{aligned}$$

Hence $\{m\} \otimes \{2, 1^2\}$ may be computed by inspection from the calculations for $\{m\} \otimes \{4\}$.

To compute $\{m\} \otimes \{1^4\}$, $\{m\} \otimes \{3, 1\}$ we modify the above method as follows:

$$\begin{aligned} \{m\} \otimes \{1^4\} &= \frac{1}{24}(t_1^4 - 6t_1^2 t_2 + 3t_2^2 + 8t_1 t_1 - 6t_4) \\ &= \frac{1}{12} \left[\frac{3}{2}(t_1^2 - t_2)^2 - t_1^4 + 4t_1 t_1 - 3t_4 \right]. \end{aligned}$$

This calculation follows at once from the results for $\{m\} \otimes \{4\}$ except for the term $\frac{1}{2}(t_1^2 - t_2)^2$. Now

$$t_1^2 = \{m\}^2 = \{2m\} + \{2m-1, 1\} + \{2m-2, 2\} + \dots + \{m, m\}$$

and by Theorem 1 we have

$$t_2 = \{2m\} - \{2m-1, 1\} + \dots + (-1)^m \{m, m\}.$$

Hence the term

$$\frac{1}{2}(t_1^2 - t_2) = \sum_v \{2m-v, v\}, \quad (v = 1, 3, \dots, m'),$$

where m' is the greatest odd integer $\leq m$. The term $\frac{1}{2}(t_1^2 - t_2)^2$ is now computed by the ordinary multiplication of S-functions.

Now

$$\{m\} \otimes \{3, 1\} = \frac{1}{12} \left[3t_1^4 - \frac{3}{2}(t_1^2 - t_2)^2 - 3t_4 \right],$$

hence this case follows by inspection from the calculations for $\{m\} \otimes \{1^4\}$.

For the remaining case $\{m\} \otimes \{2^2\}$ we have

$$\{m\} \otimes \{2^2\} = \frac{1}{12} \left[3(t_1^4 + t_2^2) - 2t_1^4 - 4t_3t_1 \right].$$

Here the coefficient of $t_1^2t_2$ is zero but the quantity t_2^2 must be calculated. We do this indirectly by making use of the results already obtained for $\{m\} \otimes \{4\}$, $\{m\} \otimes \{1^4\}$, and the following identity:

$$\frac{1}{2} \left[(t_1^2 + t_2)^2 + (t_1^2 - t_2)^2 \right] = (t_1^4 + t_2^2).$$

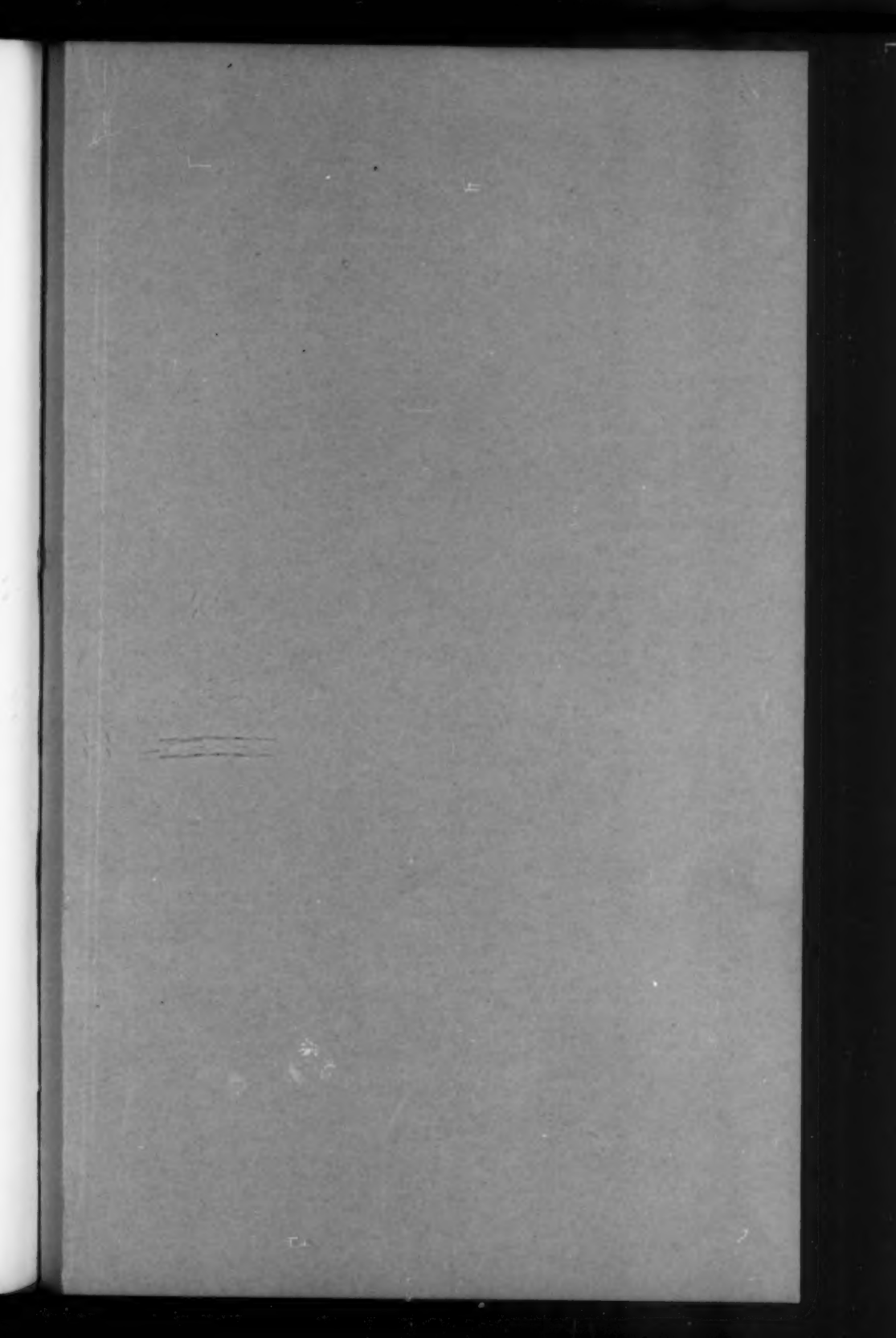
6. Conclusion. By means of the method developed here and the earlier work of Thrall, the cases $\{m\} \otimes \{2\}$, $\{m\} \otimes \{3\}$, $\{m\} \otimes \{4\}$ may be computed directly. The next case, $\{m\} \otimes \{5\}$, is considerably more complicated and does not readily lend itself to direct calculation.

The author has used this method to compute the products $\{7\} \otimes \{4\}$, $\{7\} \otimes \{2, 1^2\}$ in full, *Some Results in Littlewood's Algebra of S-functions*, thesis (microfilmed), University of Michigan, 1950. The cases $\{5\} \otimes \{4\}$, $\{6\} \otimes \{4\}$ have been computed recently by another method by Foulkes [2].

REFERENCES

1. D. G. Duncan, *Note on a formula by Todd*, J. London Math. Soc., vol. 27 (1952), 235-236.
2. H. O. Foulkes, *Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form*, J. London Math. Soc., vol. 25 (1950), 205-209.
3. D. E. Littlewood, *The theory of group characters and matrix representations of groups* (Oxford, 1950).
4. F. D. Murnaghan, *The theory of group representations* (Baltimore, 1938).
5. T. Nakayama, *On some modular properties of irreducible representations of a symmetric group*, Jap. J. Math., vol. 17 (1940), 165-184.
6. G. de B. Robinson, *On the representations of the symmetric group III*, Amer. J. Math., vol. 70, (1948), 277-294.
7. ———, *On the disjoint product of irreducible representations of the symmetric group*, Can. J. Math., vol. 1 (1949), 166-175.
8. ———, *Induced representations and invariants*, Can. J. Math., vol. 2 (1950), 334-343.
9. R. A. Staal, *Star Diagrams and the symmetric group*, Can. J. Math., vol. 2 (1950), 79-92.
10. R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math., vol. 64 (1942), 371-388.
11. R. M. Thrall and G. de B. Robinson, *Supplement to a paper of G. de B. Robinson*, Amer. J. Math., vol. 73 (1951), 721-724.
12. J. A. Todd, *A note on the algebra of S-functions*, Proc. Cambridge Phil. Soc., vol. 45 (1949), 328-334.
13. M. Zia-ud-Din, Proc. Edinburgh Math. Soc., vol. 5 (1936), 43-45.

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